On the Formation of Political Coalitions

by

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The paper analyses the process of coalition formation among political parties using game-theoretic concepts. Two different motives determining that process are distinguished: the parties' preferences over alternative policies and the politicians' desire to get into office. Based on these motives, two alternative models of coalition formation are suggested. It is shown that in situations involving only a few political parties – such as, e.g., in Germany or Austria – office-seeking considerations may generate stable coalition structures. On the other hand, if the number of parties becomes too large, stable solutions exist only under rather restrictive additional conditions. Several examples are provided illustrating the theoretical implications of the analysis. (JEL: D 72, C 71)

1. Introduction

The process of government formation beginning with a particular election outcome and resulting in a particular government, mostly consisting of a coalition among different political parties, is certainly one of the fundamental processes in European parliamentary democracies. As LAVER and SCHOFIELD [1990, 89] put it, "understanding how a given election result leads to a given government is ... simply the most important substantive projects in political science." The present paper seeks to contribute to our understanding of the formation of political coalitions using game-theoretic concepts, and thus relating the subject to the economic theory of cooperative decision making. The problem of coalition formation has been one of the central concerns of game theory since VON NEUMANN and MORGENSTERN's [1953] fundamental *Theory of Games and Economic Behaviour*. The application of game theoretic concepts to political processes and, in particular, to the formation of government coalitions also has a long tradition since RIKER's [1962] classic monograph *The Theory of Political Coalitions*. In the literature, one can distinguish two fundamentally different approaches to that problem. The first, starting with Riker's account, focusses on the *office-seeking* interpretation of coalition formation (see also GAMSON [1961], and LAVER and SCHOFIELD [1990] for an extensive survey). The

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basic assumption here is that politicians are above all else motivated by the desire to get into office. More precisely, suppose that after a political election a certain government coalition comprising different political parties has been formed. Also, assume that the seats in the government's cabinet are allocated proportionally to the share of votes each government party has received (see footnote 2 below). Then the basic assumption is that parties care above all else about the number of seats they receive in the cabinet, or equivalently, about their respective *relative power* within the government coalition. Here, we have identified a party's relative power with its relative weight within a coalition, i.e. with the number of votes it received divided by the total number of votes for all coalition members. Henceforth, the case in which parties care *only* for their relative power is referred to as the case of *purely office-seeking* behaviour. Note that one may think of this special case as a constant sum game. Indeed, there is a fixed price, the total number of seats in the cabinet, and every portion of this prize won by one actor must be lost by another. The assumption of office-seeking behaviour certainly captures an important aspect of the motives for coalition formation. On the other hand, the assumption of *purely* office-seeking behaviour seems to be rather extreme, probably over-emphasising that aspect.

A second fundamental motive in the process of coalition formation among political parties derives from the parties' *policy preferences*. Indeed, there can be no doubt that the parties' evaluation of alternative coalitions will to a considerable extent also depend on their preferences over the alternative policy programs advocated by the different coalition members. Based on this observation, the second approach in the literature, starting with PLOTT'S [1967] seminal work, has concentrated on the role of policy preferences as the main behavioural motive in the context of political decisions (for further references, see LAVER and SCHOFIELD [1990, ch. 5, footnote 28]). Unfortunately, the basic result obtained in this literature is rather negative. Indeed, in various specific frameworks it has turned out that, under majority rule, policy bargaining between different political parties almost never leads to stable coalition structures (see, e.g., SCHOFIELD [1983]). It is worth noting, however, that in this literature office-seeking considerations of political parties play no role whatsoever. Indeed, it is quite surprising that only a few attempts have been made to combine the two different motives identified by the literature. [1]

The main purpose of the present study is to provide models of coalition formation which explicitly take into account both, the parties' policy preferences as well as their members' desire to get into office. Although it may be intuitively clear that both motives do play an important role in the process of coalition formation, it is worth emphasising that there is also some empirical evidence for this. As an example, consider the situation after the national

[1] Among the few exceptions are the models of AXELROD [1970] and DE SWAAN [1973]. However, in their models policy preferences are only *implicitly* taken into account.

elections ("Nationalratswahlen") in 1994 in Austria (the situation being essentially the same as in the years 1986 and 1990). The Social Democrats (SPÖ) proved to be strongest party without gaining the absolute majority of seats in the parliament. Furthermore, the "Volkspartei" (ÖVP) turned out to be the second biggest party, followed by the "Freiheitliche Partei" (FPÖ). The distribution of votes was such that a coalition of the ÖVP with the FPÖ would have gained the absolute majority of seats in the parliament. As is well known, the ÖVP chose to form a coalition with the Social Democrats, a decision obviously due to policy preferences. However, in terms of office-seeking considerations a coalition with the FPÖ would have been more advantageous. Due to greater relative power in a coalition with a smaller coalition partner, the ÖVP would probably have gained some additional seats in the cabinet and, moreover, the chance to provide the chancellor. Indeed, although the ÖVP refrained from a coalition with the FPÖ, the incentive to do so can be clearly identified in the political speeches and discussions which followed the election result.

The basic game-theoretic solution concept to be applied in the following analysis is that of the *core* of a cooperative game. A possible *government coalition* is a coalition which jointly achieves at least half of the total number of votes. Each party's preference among different possible coalitions is assumed to depend on its policy preferences as well as on its respective relative power, or relative weight, within a coalition. Our analysis retains the fundamental assumption that, given a certain government coalition, each government party's share of the seats in the cabinet is proportional to the share of votes it received. [2] Consequently, relative power is identified with the number of seats in the cabinet. In particular, relative power is pre-determined by the election result and not subject to any further bargaining process among coalition members. Clearly, it is assumed that *ceteris paribus* each party prefers coalitions which offer greater relative power. In contrast to relative power, the policy program realised by a certain government coalition is of course subject to a bargaining process among the coalition members. Indeed, here lies the fundamental difficulty for any theory of coalition formation: In order to determine the parties' preferences among different coalitions, one has to specify each party's expectations regarding the outcome of the policy bargaining process within a given coalition. But as the literature has repeatedly demonstrated, there is no completely satisfactory model of policy bargaining within coalitions. Indeed, at least under majority rule one cannot expect a stable outcome of the policy bargaining process within coalitions. The present paper does not attempt to provide a general account of policy bargaining, let alone a *general* theory of coalition formation. Indeed, such a general theory seems to be far ahead of the present state of economic research (see, however, PELEG [1980], [1981]). Rather, we offer two alternative approaches each of which emphasises one particular

[2] This assumption is strongly confirmed by the empirical study of BROWNE and FRANKLIN [1973] based on the theoretical work of GAMSON [1961].

empirically relevant aspect of the problem. Hence, the two distinct approaches suggested below do not compete for the status of the "right" model of coalition formation among political parties. Nevertheless, either model may be fruitfully applied to different empirical circumstances, thus providing at least partial insight into the problem of coalition formation.

The first approach is based on the assumption that in any given coalition the biggest included party can always enforce the realisation of its most preferred policy program. In general, this corresponds not to majority rule but to simple *plurality voting* within coalitions. [3] Clearly, this assumption might not be applicable under all circumstances. However, in many cases the implied simplification may serve at least as a first approximation to the real life process of policy bargaining. Our second approach avoids the problem of determining a specific outcome of policy bargaining by introducing a measure of "ideological distance" between different parties. This approach generalises the models of AXELROD [1970] and DE SWAAN [1973], assuming that parties prefer coalitions whose members are "ideologically close."

1.1 Overview of the Main Results

In section 2, it is shown that in the case of purely office-seeking behaviour the core is with some negligible exceptions always non-empty and single-valued. It always consists of the *minimum* winning coalition, i.e. the coalition with the smallest total number of votes which nevertheless gains the absolute majority of seats in the parliament. Although this result seems to be known in the literature, we are not aware of a rigorous proof using the core concept. The result by itself is not ultimately satisfactory, for it rests on the somewhat unrealistic assumption that parties are only guided by their desire to get into office. On the other hand, the underlying argument hints at the conditions which are sufficient for possibility results, i.e. the existence of stable government coalitions, in a more general framework.

Policy preferences are introduced in section 3 which is divided into two parts. In the first part (subsection 3.1), we investigate the scenario in which the biggest government party is always able to enforce the realisation of its most preferred policy program. Although we also provide a general analysis of this scenario, the most interesting results apply to the case where there are only a few parties. For many applications this will also be the most relevant case. For instance, the political situation in Germany and Austria has been essentially a three party system for quite long a period. Since the rise of the "Green parties" this has tranformed into a four party system in Germany. In Austria, the present situation is best described as a five party system comprising SPÖ, ÖVP, FPÖ

[3] Note, however, that the outcomes of simple plurality voting and the majority rule coincide whenever there is a "dominant" player (i.e., "dominant" party) as assumed, e.g., in PELEG [1981].

(nowadays called "F"), "Die Grünen" and the Liberals. Consequently, the main focus of our analysis is on the cases of three, four and five parties.

In the three party case, the core is never empty with the smallest party acting as the *pivotal party*, i.e. the party that essentially determines which coalition will be formed. In particular, it turns out that a coalition between the two biggest parties can never be a stable outcome of the coalition formation process. In the four party case, a core solution may not exist. On the other hand, if the two smallest parties are both purely office-seeking, core existence is again established. Observe that in the present context the assumption of purely office-seeking behaviour for *small* parties is not implausible. Indeed, if policy is decided upon within coalitions by plurality voting the two smallest parties have no direct influence on realised policy. Given this, it seems reasonable to assume that relative power plays a greater role for small parties. Perhaps the most interesting case from a theoretical point of view is the case of five parties. It is shown that the core always consists of a single stable coalition provided that the three smallest parties are office-seeking in the above sense. Nevertheless, the unique core solution is not always the minimum winning coalition as in the case in which all parties are purely office-seeking. Hence, it is in this case that the departure of our approach from previous studies becomes most evident.

In the general case, with an arbitrary number of parties much stronger conditions are needed in order to guarantee the existence of stable government coalitions. Specifically, it is shown that stable solutions exist whenever, in addition to all small parties being office-seeking in the sense explained above, the medium parties are "strongly" office-seeking, i.e. office-seeking regardless of whether or not they can directly influence realised policy. Clearly, the latter assumption has much less intuitive appeal. However, without strong assumptions the core will in general be empty. The fact that a large number of parties makes the formation of stable government coalitions more difficult can also be observed in reality. A prominent example is the multi-party system in Italy which is notorious for its instability. The theoretical reasons for this are very similar to those which are responsible for the negative results in the context of policy bargaining under majority rule.

Subsection 3.2 introduces a measure of "ideological distance" between different political parties and suggests a certain form in which the parties' preferences among coalitions depend on that measure. The advantage of such an approach is that there is no need to specify a particular outcome of the policy bargaining process within coalitions. All that is implicitly assumed is that, for each party involved, this outcome will be more favourable if the coalition partners are ideologically close. Again, the most interesting result applies to the case of a few parties. Specifically, it is proved that in the three party case there is always a stable outcome of the coalition formation process. In contrast to the previous model, our second approach allows for a coalition between the two biggest parties as the only stable government coalition in the three party case. Furthermore, a sufficient condition for core solutions in the general case with an arbitrary number of parties is provided. As in our previous model, the general sufficient condition is rather restrictive, requiring the distribution of distances between parties to be close to the uniform distribution. Indeed, in this case, office-seeking considerations again become decisive.

From a theoretical point of view, the main conclusion of our analysis is hence as follows. In situations involving only a few political parties, office-seeking considerations may play a central role for the formation of stable government coalitions. With many parties, on the other hand, office-seeking behaviour is not sufficient for stable coalition structures, at least unless one assumes rather extreme forms of such behaviour.

2. *The Case of Purely Office-Seeking Parties*

In this section we focus on the assumption that, in the process of the formation of a government coalition, political parties are motivated *only* by office-seeking considerations. Specifically, it will be assumed that – given their respective number of votes – parties maximise their relative weight within a coalition. As explained in the introduction, this can be interpreted as the assumption that parties simply maximise the number of seats in the cabinet they receive. It has already been noted that we do not consider this to be an ultimately satisfactory assumption. However, in the context of the present paper it conveniently allows us to introduce our basic concepts and to derive some fundamental results which will then serve as reference points for our subsequent analysis.

Consider n different political parties denoted by $1, 2, \ldots, n$ and let N denote the set of all those parties, i.e. $N = \{1, 2, \ldots, n\}$. It is assumed that, after a political election in a large population, each party is endowed with a certain number of votes. For notational convenience, it is assumed that the total size of the population is 1, so that for each $i \in \{1, \ldots, n\}$, the number of votes for party i may be represented by a real number α_i with $0 \leq \alpha_i \leq 1$, referred to as the *weight* of party i. Obviously, parties with no votes play no role in our static one-shot analysis, hence it is always assumed that all weights α_i are strictly positive. Clearly, one has $\sum_{i=1}^{n} \alpha_i = 1$. Without loss of generality, parties are always indexed in descending order, i.e. $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. Throughout this paper it is assumed that a *government* must comprise more than half of the votes.[4] First, suppose that $\alpha_1 > \frac{1}{2}$, hence suppose that party 1 possesses the

[4] It should be noted that this requirement is not as straightforward as it may seem. Indeed, some authors have suggested that a properly functioning government needs some extra "safety seats" in the parliament, in addition to the bare majority of seats (see, e.g., LAVER and SCHOFIELD [1990, 94]). However, most of the results of our analysis do not depend on the choice of the critical proportion of votes (seats in parliament), as long as this number is at least $\frac{1}{2}$. Therefore, the present paper abstracts from the issue of "safety seats" and concentrates on the simplest case where the critical proportion of seats is exactly $\frac{1}{2}$.

absolute majority of votes. In this case, party 1 is able to govern alone. Consequently, there is no incentive to coalesce with any other party, a fact which is well supported by the empirical evidence. Hence, in what follows we will be concerned mainly with the case where $\alpha_1 < \frac{1}{2}$.[5] In this case, parties necessarily have to cooperate in order to form a functioning government coalition. The set of possible government coalitions is $\mathscr{W} = \{S \subseteq N: \sum_{i \in S} \alpha_i > \frac{1}{2}\}$. In the literature, a coalition $S \in \mathscr{W}$ is referred to as a *winning coalition*. The basic question to be asked is: *which* winning coalition will form? Clearly, this will depend on the respective payoffs for each party. Under the present assumption that parties care only for their relative weight in a coalition, these payoffs are easily described: Suppose that a winning coalition S forms. Then the payoff $v(S)_i$ for party i is given by

$$v(S)_i = \begin{cases} \dfrac{\alpha_i}{\sum_{j \in S} \alpha_j} & \text{if } i \in S \\ 0 & \text{if } i \notin S\,. \end{cases} \tag{1}$$

Hence, for each $S \subseteq N$ one may associate a *feasible utility* vector $v(S) := (v(S)_i)_{i \in N} \in \mathbb{R}^N$ with the convention that $v(S)_i = 0$ for all $i \in N$ if S is not winning. Consequently, the specification (1) defines a *cooperative game with non-transferable utility* (*NTU game*) among the n parties. Note that there is a one-to-one correspondence between the set of winning coalitions and the set of all possible non-zero feasible utility vectors.[6] The basic solution concept to be used in the subsequent analysis is that of the *core*. A feasible utility vector $v(S)$, or equivalently, the corresponding coalition S, is in the core of the underlying NTU game if and only if there is no non-empty coalition T such that for all $i \in T$, $v(T)_i \geq v(S)_i$ with at least one inequality being strict. First, observe that for some distributions of weights the core of the corresponding NTU game may be empty.

Example 1. Let there be three parties with corresponding weights $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. In this scenario, the set of all winning coalitions is

$$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\,.$$

It is easily verified that any of these coalitions is dominated by some other coalition in that set. Indeed, the feasible utility vectors for these coalitions are

[5] As will become clear, the case $\alpha_1 = \frac{1}{2}$ may be neglected in our analysis.

[6] In the standard theory of NTU games the set of feasible utility vectors is usually assumed to be comprehensive. This amounts to assuming that if $v \in \mathbb{R}^N$ is feasible for coalition S then so should be any vector $w < v$, where $w < v$ means $w_i \leq v_i$ for all i with at least one strict inequality. Intuitively, such a condition corresponds to the assumption of "free disposal of utility." While such an assumption could formally be incorporated in our analysis without difficulty it would not affect any result.

given by

$$v(\{1, 2\}) = (1/2, 1/2, 0),$$
$$v(\{1, 3\}) = (1/2, 0, 1/2),$$
$$v(\{2, 3\}) = (0, 1/2, 1/2),$$
$$v(\{1, 2, 3\}) = (1/3, 1/3, 1/3).$$

Obviously, the grand coalition $\{1, 2, 3\}$ is dominated by all other winning coalitions, and each two-party coalition is dominated by all other two-party coalitions. Hence, it follows that the core is empty.

Remark. As a response to example 1, one might argue for the following alternative definition of the core. Say that a coalition S is in the core if and only if there exists no non-empty coalition T such that for all $i \in T$, $v(T)_i \geq v(S)_i$, with $v(T)_i > v(S)_i$ for some $i \in T \cap S$. Note that this definition of the core is weaker than the previous definition in the sense that any coalition in the core according to the previous definition is also in the core according to this definition. The reason is that the strict inequality is now required to hold for some party contained in *both T and S*. It is easily verified that, according to the alternative definition, the core in example 1 would consist of coalitions $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. Moreover, one can show that according to this definition the core is never empty in our present context of purely office-seeking parties. However, as example 1 demonstrates, the problem of *non-existence* of the core is then simply transformed into the problem of *multiple* core solutions. Indeed, whenever the core is empty according to our previous definition there will be several coalitions in the core according to the alternative definition. Although the alternative definition of the core has the advantage that the set of solutions depends upper hemicontinuously on the exogenous variables, the more demanding definition that we have chosen seems to be better suited for our purpose since it permits only *robust* solutions.

The fact that the core may be empty (or, according to the definition in the remark: non-single valued) is, however, not a serious problem in our present context. Indeed, due to our assumption that voters form a *large population* this is in a precise sense a rare phenomenon. It will be shown below that *generically* the core is not only non-empty but consists of exactly one element. Here, the term "generically" refers to the fact that in the space of all possible NTU games derived from some distribution of weights by means of (1), the set of those games with empty core is a closed set of measure zero. More precisely, let the population of voters be represented by the unit interval [0, 1]. Then any point in the $(n-1)$ dimensional unit simplex $S^{n-1} := \{\alpha \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \alpha_i = 1\}$ is a possible distribution of weights. Consequently, the set of all NTU games induced by (1) may be parameterised by the points of S^{n-1}. A property of a NTU game is said to hold generically whenever it holds except for a closed set of Lebesgue measure zero in the unit simplex.

In order to derive the generic existence of the core, it is convenient to introduce the following terminology. Say that a winning coalition S is *minimal* if and only if the substraction of any party from S would make the remaining coalition no longer a winning one (see LAVER and SCHOFIELD [1990], and also VON NEUMANN and MORGENSTERN [1953]). Formally, S is a minimal winning coalition if and only if $S \in \mathscr{W}$ and, for all $i \in S$, $\sum_{j \in S} \alpha_j - \alpha_i < \frac{1}{2}$. For instance, in example 1 above the minimal winning coalitions are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. Observe, that a non-minimal winning coalition can never be in the core if all weights are positive. Indeed, let $S \in \mathscr{W}$ be non-minimal, i.e. suppose that, for some $i \in S$, one has $\alpha_i > 0$ and $\sum_{j \in S} \alpha_j - \alpha_i > \frac{1}{2}$. Clearly, in this case

$$v(S \setminus \{i\})_j > v(S)_j \quad \text{for all} \quad j \in S \setminus \{i\},$$

hence $S \setminus \{i\}$ dominates S. Consequently, the core can only consist of minimal winning coalitions. However, even generically, minimality is not sufficient for a coalition to be in the core. Consider the following example.

Example 2. Let there be three parties with weights $\alpha_1 = 0.40$, $\alpha_2 = 0.35$ and $\alpha_3 = 0.25$. As in the previous example, the set of all minimal winning coalitions is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. However, for instance the coalition $\{1, 2\}$ is not in the core of the corresponding NTU game. The feasible utility vectors for the three minimal winning coalitions are given by

$$\begin{aligned} v(\{1, 2\}) &= (0.4/0.75,\ 0.35/0.75,\ 0) = (0.533,\ 0.467,\ 0)\,, \\ v(\{1, 3\}) &= (0.4/0.65,\ 0,\ 0.25/0.65) = (0.615,\ 0,\ 0.385)\,, \\ v(\{2, 3\}) &= (0,\ 0.35/0.6,\ 0.25/0.6) \ = (0,\ 0.583,\ 0.417)\,. \end{aligned}$$

From this it is clear that $\{1, 2\}$ is dominated by both $\{1, 3\}$ and $\{2, 3\}$, whereas $\{1, 3\}$ is dominated by $\{2, 3\}$. On the other hand, the coalition $\{2, 3\}$ is not dominated by any other winning coalition, hence in this example the core consists of the coalition $\{2, 3\}$.

The reason for coalition $\{2, 3\}$ to be the unique solution in example 2 is of course the fact that it is the coalition with the smallest total weight among all winning coalitions. Following the terminology of LAVER and SCHOFIELD [1990], we will say that a minimal winning coalition is a *minimum* winning coalition if and only if there is no winning coalition with strictly less total weight, i.e. S is a minimum winning coalition if and only if for all $T \in \mathscr{W}$, $\sum_{j \in T} \alpha_j \geq \sum_{j \in S} \alpha_j$.[7] The

[7] It should be noted that other authors have used different terms. For instance, GAMSON [1961] refers to the minimum winning coalitions as the *cheapest* winning coalitions. Sometimes minimum winning coalitions are also referred to as *bare majority* coalitions. On the other hand, RIKER [1962] uses the term "minimum coalition" for what we have called "minimal coalition."

following lemma demonstrates the generic uniqueness of a minimum winning coalition.

Lemma 1: Generically in the space of all NTU games derived from some distribution of weights according to (1), there is a unique minimum winning coalition.

Proof: In order to verify this it will be shown that the set A_0 of weights $(\alpha_i)_{i \in N}$, for which there exist subsets S, $T \subseteq N$, $S \neq T$, such that

$$\sum_{j \in S} \alpha_j = \sum_{j \in T} \alpha_j, \tag{2}$$

is a closed set of $(n-1)$ dimensional measure zero. Obviously, equation (2) defines a linear subspace of dimension $m < n - 1$ in the $(n-1)$ dimensional unit simplex. Each such subspace is closed and has $(n-1)$ dimensional Lebesgue measure zero. Since A_0 is a finite union of such linear subspaces, it must also be a closed set of measure zero. *Q.E.D.*

We are now ready to state the main result of this section.

Theorem 1: Generically in the space of all NTU games derived from a distribution $(\alpha_i)_{i \in N}$ of weights according to (1), the core is non-empty and consists of the unique minimum winning coalition.

Proof: In order to prove this claim, let S be the (generically) unique minimum winning coalition according to lemma 1. Consider any other coalition $T \subseteq N$. We have to verify that $v(T)_j \leq v(S)_j$ for all $j \in T$, or $v(T)_i < v(S)_i$ for some $i \in T$. If T is not winning, one has $v(T) = 0$, hence $v(T)_j \leq v(S)_j$ for all $j \in T$. If T is winning, the set $T \cap S$ is non-empty, i.e. there must exist some parties which are contained in both coalitions. Let j be any such party. Since S is the unique minimum winning coalition the total weight of S is less than the total weight of T. Consequently, for any $j \in T \cap S$, $v(T)_j < v(S)_j$, hence S cannot be dominated by T. This shows that S is indeed an element of the core.

On the other hand, suppose that T is a winning coalition but not the minimum winning coalition. Clearly, for each member j of the minimum winning coalition S one has $v(S)_j > v(T)_j$, so that S dominates T. Hence, T is not in the core. *Q.E.D.*

It must be emphasised that theorem 1 is by no means a new result. Indeed, similar results can be found in the literature since the early work on coalition formation in the 1960's (see, e.g., LAVER and SCHOFIELD [1990, ch. 5] for a survey). However, to the best of our knowledge, theorem 1 above is the first rigorous account using the core concept and emphasising the generic nature of the result.

3. Including Policy Preferences

In this section, we abandon the restrictive assumption that parties are *only* motivated by office-seeking considerations, i.e. by their relative power in a coalition. Indeed, there can be no doubt that in real life decisions the parties' preferences over *expected policy* will also play an important role in the formation of a government coalition. In order to include this aspect in our analysis, we will assume henceforth that the parties' preferences are defined on a set $X \times [0, 1]$, where X is a *policy space* and the elements of $[0, 1]$ represent as before the parties relative weight in a coalition. The new ingredient is thus the policy space X. Intuitively, it consists of all logically possible overall political programs. In particular, for any party $i \in N$ there is associated one and only one element $x_i \in X$ representing the political program advocated by party i. Throughout, it is assumed that $x_i \neq x_j$ for all distinct $i, j \in N$. Hence, it is assumed that no two parties advocate the *same* political program. This seems to be a very plausible assumption. Indeed, a complete coincidence of political views and aims between different parties is most unlikely to happen. Observe that this assumption implies that the policy space X must have at least n elements. No further assumptions on the structure of X will be needed.

The preferences of party i over the space $X \times [0, 1]$ are denoted by $\succeq_i$. For any $x, y \in X$ and any $\xi, \eta \in [0, 1]$, the statement $(x, \xi) \succeq_i (y, \eta)$ thus stands for "party i weakly prefers the (realised) policy program x while having the relative weight ξ in the governing coalition to the policy program y while having the relative weight η in the government." Throughout, $\succeq_i$ is assumed to be transitive. Furthermore, on the subset $\{x_1, \ldots, x_n\} \times [0, 1]$ the preference relation $\succeq_i$ is assumed to be complete. Hence, each party is able to compare for any given pair of relative weights the policy programs actually advocated by some party. Throughout, the strict preference for the parties' own policy program is required to be *persistent* in the following sense. For all $i \in N$, all $\xi \in [0, 1]$, and all $j \neq i$, $(x_i, \xi) \succ_i (x_j, \xi)$. Hence, each party strictly prefers its own policy program to any other advocated policy program at any fixed level of relative power.

The following property is a monotonicity condition in the second component ("monotonicity in relative power"):

$$\text{(M)} \qquad \xi > \eta \Rightarrow (x, \xi) \succ_i (x, \eta), \quad \text{for all} \quad i \in N,\ x \in X, \text{ and all } \xi, \eta \in [0, 1].$$

This seems to be a rather innocuous condition, requiring that for any fixed policy each party strictly prefers having more relative power. Note that in combination with our other assumptions, condition (M) in particular implies that for all $i \in N$, all $\xi, \eta \in [0, 1]$, and all $j \neq i$,

$$\xi > \eta \Rightarrow (x_i, \xi) \succ_i (x_j, \eta).$$

Thus, each party strictly prefers it own policy program at a higher level of relative power to any other policy program actually advocated at less relative power.

As already discussed in the introduction, a fundamental difficulty in deriving from the parties' preferences over policies the set of undominated coalitions, i.e. those coalitions which are likely to be formed, consists in determining which policy will be realised given a certain coalition. Clearly, whether a party prefers participation in one or the other coalition crucially depends on the expectation of the realised policy program given each of the relevant coalitions. Hence, it seems that one would need a theory of *policy bargaining* among coalition partners.[8] However, the lack of a general theory of policy bargaining between coalition partners necessitates particular assumptions which may not be applicable under all circumstances. In the following, we offer two alternative approaches each of which captures different relevant features of the real life process of coalition formation.

3.1 Policy Bargaining by Plurality Voting

In this subsection we examine the special case in which the realised policy program is always determined by the biggest coalition member. Hence, given a winning coalition, the realised policy program is the one which is most preferred by the party with greatest relative power within the coalition. Certainly, we do not claim that this is plausible under all circumstances. In particular, this assumption leaves no room for possible *compromises* among coalition partners. However, there are many countries in which a single person, usually the chancellor or prime minister, has a strong influence on actual policy. Since the government leader usually belongs to the biggest coalition party, our present assumption can under such circumstances serve at least as a first approximation to the real life process of policy bargaining.

It is important to realise that a dictatorial leadership of the biggest coalition partner may well be the outcome of a social choice process. Indeed, suppose that the realised policy program is decided upon within a coalition according to the *plurality rule*. This means that any member of the parliament belonging to a certain winning coalition, i.e. any member of a government party, has exactly one vote and casts the vote for his or her most preferred policy program. The realised policy program is that which is voted for most often. Assuming that any member of the parliament prefers the policy program advocated by the party he or she belongs to, the plurality rule of course amounts to assuming that any coalition is ruled by its biggest member. Hence, our present assumption may be referred to as the assumption of policy bargaining by plurality voting.

[8] A note on terminology: One may, of course, interpret any formation of a particular coalition as the outcome of a "policy bargaining." However, what we mean here by the term "policy bargaining" is only the process that results in the implementation of a certain policy *given* a fixed government coalition.

In formal terms, the assumption that within a winning coalition policy is determined by plurality voting may be described by the following *outcome function* $w\colon \mathscr{W} \to X \times \mathbb{R}^N$. For any given $S \in \mathscr{W}$, let i_S^* denote the (generically unique) biggest party in S. Given that the winning coalition S has formed, for each party i the outcome $w(S)_i \in X \times [0, 1]$ is given by

$$w(S)_i = \begin{cases} \left(x_{i_S^*}, \dfrac{\alpha_i}{\sum\limits_{j \in S} \alpha_j}\right) & \text{if } i \in S \\ (x_{i_S^*}, 0) & \text{if } i \notin S . \end{cases} \tag{3}$$

A winning coalition S is in the core of the underlying NTU game if and only if there is no winning coalition T such that for all $i \in T$, $w(T)_i \succeq_i w(S)_i$ with at least one preference being strict.

The following result shows that the core is always a (possibly empty) subset of the set of all minimal winning coalitions.

Lemma 2: Suppose that condition (M) is satisfied. Then, a non-minimal winning coalition is never in the core of an NTU game derived from a distribution of positive weights according to (3).

Proof: In order to prove this, let S be a non-minimal winning coalition. Hence, there exists $i \in S$ such that $\sum\limits_{j \in S} \alpha_j - \alpha_i > \frac{1}{2}$. Without loss of generality, suppose that i is the smallest party in coalition S. In particular, the realised policy program given coalition S is not equal to x_i. Consequently, one has $x_{i_S^*} = x_{i^*_{S \setminus \{i\}}}$, and by condition (M), $w(S \setminus \{i\})_j \succ_j w(S)_j$ for all $j \in S \setminus \{i\}$. Hence, S is dominated by $S \setminus \{i\}$ and can therefore not be a core solution. *Q.E.D.*

Before we further investigate the scenario with plurality voting in general, we concentrate first on situations where there are only a few different political parties, as is the case in, e.g., Germany or Austria.

3.1.1 The Case of Three Parties

Suppose that there are only three parties with a positive number of votes. Assuming voting weights to be in decreasing order, one has generically, $\alpha_1 > \alpha_2 > \alpha_3$. As before denote by $x_1, x_2, x_3 \in X$ the most preferred policy programs of these parties. If $\alpha_1 > \frac{1}{2}$, then no matter what the preferences of the other parties are, party 1 will stand alone in forming a government. Hence, suppose that $\alpha_1 < \frac{1}{2}$. In this case, the minimal winning coalitions are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. Note that by lemma 2 the grand coalition $\{1, 2, 3\}$ is dominated by both $\{1, 2\}$ and $\{1, 3\}$. Observe also that unlike in the case of purely office-seeking parties, the grand coalition $\{1, 2, 3\}$ might not be dominated by $\{2, 3\}$. Indeed, knowing that its own policy program has no chance of being realised, party 3 may well prefer the grand coalition to the coalition $\{2, 3\}$. The reason may be a strong preference for x_1 over x_2 overriding the disadvantage of less relative power in the grand coalition.

It is also easily verified that condition (M) implies that $\{1, 2\}$ is dominated by $\{1, 3\}$. The reason is that since party 1 can in both coalitions ensure its most preferred program x_1 to be realised, it strictly prefers $\{1, 3\}$ due to the greater relative power. Clearly, party 3 also prefers $\{1, 3\}$ to $\{1, 2\}$. Consequently, in the three party case the only possible core coalitions are $\{1, 3\}$ or $\{2, 3\}$. More precisely, one has the following result.

Fact 1: Suppose that $n = 3$ and that condition (M) holds. Generically, the core of the NTU game corresponding to (3) is non-empty and single-valued. It consists either of coalition $\{1, 3\}$ or of coalition $\{2, 3\}$.

Indeed, it is easily verified that the core consists of coalition $\{1, 3\}$ if and only if party 3's preference satisfies

$$\left(x_1, \frac{\alpha_3}{\alpha_1 + \alpha_3}\right) \succ_3 \left(x_2, \frac{\alpha_3}{\alpha_2 + \alpha_3}\right).$$

Similarly, the core consists of coalition $\{2, 3\}$ if and only if

$$\left(x_1, \frac{\alpha_3}{\alpha_1 + \alpha_3}\right) \prec_3 \left(x_2, \frac{\alpha_3}{\alpha_2 + \alpha_3}\right).$$

Consequently, in the three party case the smallest party 3 is the *pivotal party* in the sense that it alone determines which coalition will form. This result seems to be well supported by the empirical evidence. A prominent example is the distinctive role of the Liberal Democrats (F.D.P.) in Germany after World War II. Neglecting the very small influence of some minor parties, the political situation in Germany until 1983 was essentially a three party system. In 1982, the smallest party, the Liberal Democrats, indeed clearly demonstrated their strategic position as pivotal party when they quit the coalition with the Social Democrats in order to form a government coalition with the Christ Democrats. (Of course, the distinctive strategic position of the F.D.P. had been apparent also before 1982.) Note that, theoretically, there might also be the case where party 3 is indifferent between coalitions $\{1, 3\}$ and $\{2, 3\}$. However, under very mild regularity assumptions this can happen only on a closed set of measure zero in the unit simplex.[9] Hence, this is a non-generic case.

The result that in the three party case of our model a *big parties' coalition* – meaning a coalition between the two biggest parties ("Große Koalition") –

[9] Observe, that by condition (M), for any fixed α_3^0, there exists at most one distribution $(\alpha_1, \alpha_2, \alpha_3^0)$ such that party 3 is indifferent between coalitions $\{1, 3\}$ and $\{2, 3\}$.

cannot occur is remarkable and needs perhaps some further comment. Although this result in a way reflects the often and widely expressed aversion to big parties' coalitions in general, it must be admitted that, as an empirical fact, such coalitions have nevertheless occurred in the past, notably in Austria. First, it is emphasised that a highly stylised model such as that used in the present analysis must neglect features of real life decision processes which may become relevant under certain circumstances. Indeed, it cannot be excluded that features outside the scope of the present analysis have caused the occurrence of big parties' coalitions. However, there is also an explanation of the phenomenon of big parties' coalitions within our theoretical framework. As has been suggested in the literature, an absolute majority of seats by a very small margin might be too uncomfortable a basis for the enterprise of ruling a whole country (see footnote 4). Hence, if one requires a government to possess at least some proportion $a > \frac{1}{2}$ of seats in the parliament, a big parties' coalition may be the only minimal winning coalition if the remaining parties are very small. We note that, in many cases where big parties' coalitions have actually formed in the past, the remaining parties were indeed relatively small. For a different explanation of the occurrence of big parties' coalitions even when not all remaining parties are small, see subsection 3.2 below.

3.1.2 The Case of Four Parties

The situation with four parties is more complex than the three party case. For instance, the core may be empty in this case. However, under relatively mild additional assumptions, generic existence and single-valuedness of the core can again be established. Consider four parties with weights $\frac{1}{2} > \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. By lemma 2, we may concentrate in the following analysis on the set of minimal winning coalitions. We distinguish two cases.

Case 1: $\alpha_2 + \alpha_3 > \frac{1}{2}$. Hence, the coalition $\{2, 3\}$ is winning. It is easily verified that in this case the set of all minimal winning coalitions consists of $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. Consequently, the situation is similar to the three party case with party 3 as the pivotal party. Indeed, it is easily verified that, generically, the core consists either of $\{1, 3\}$ or of $\{2, 3\}$, depending on 3's preferences.

Case 2: $\alpha_2 + \alpha_3 < \frac{1}{2}$. Hence $\{2, 3\}$ is not winning. In this case, the set of all minimal winning coalitions is

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}.$$

Again, we distinguish two cases.

Case 2a: $\alpha_1 < \alpha_2 + \alpha_3$. In this case, the core may be empty. Indeed, suppose that regardless of their respective relative power, party 3 has a strict preference for policy program x_1 over x_2, whereas party 4 strictly prefers x_2 over x_1. It is easily established that under these assumptions any minimal winning coalition is dominated by some other minimal winning coalition, hence the core is empty. However, one may argue that party 4's strict preference for coalition $\{2, 3, 4\}$ over coalition $\{1, 4\}$ as just assumed is not plausible under all circumstances. Indeed, knowing that its own policy program has no chance of being realised, it may well be that party 4's preference between these coalitions is determined by the respective relative power it has in each of these coalitions, in which case party 4 would opt for coalition $\{1, 4\}$. According to this idea, we will say that party i is *office-seeking* (given the weights α_1 to α_4) if and only if, among all coalitions in which i is not the biggest party, greater relative power for i implies a strict preference. Formally, for each party $i \in N$, we will consider the following condition ("party i is office-seeking"):

$$\text{OS}(i) \qquad \frac{\alpha_i}{\sum_{j \in S} \alpha_j} > \frac{\alpha_i}{\sum_{j \in T} \alpha_j} \Rightarrow \left(x_{i_S^*}, \frac{\alpha_i}{\sum_{j \in S} \alpha_j}\right) \succ_i \left(x_{i_T^*}, \frac{\alpha_i}{\sum_{j \in T} \alpha_j}\right),$$

$$\text{for all } S, T \subseteq N \text{ with } i \neq i_S^*, i_T^*.$$

In general, condition OS does not seem to be a very strong condition. Observe, however, that the intuitive strength of condition OS crucially depends on the distribution of weights. If, for instance, two coalitions in which i is contained but not the biggest party have almost the same total weight, condition OS(i) becomes less plausible. Nevertheless, in a general analysis a condition such as OS seems to be necessary to insure generic core existence.

Given that party 4 is office-seeking, i.e. given condition OS(4), it is easily verified that the core in case 2a is non-empty and consists of the coalition $\{1, 4\}$.

Case 2b: $\alpha_1 > \alpha_2 + \alpha_3$. Again, without further assumptions the core may be empty. However, if parties 3 and 4 are office-seeking it is easily established that the only core solution is the coalition $\{2, 3, 4\}$.

Summarising, one obtains the following result for the four party case.

Fact 2: Suppose that $n = 4$ and that condition (M) holds. Furthermore, assume that the two smallest parties are office-seeking, i.e. assume OS$(3, 4)$. Then, the core of the NTU game corresponding to (3) is generically non-empty and single-valued. It consists either of coalition $\{2, 3\}$, if $\alpha_2 + \alpha_3 > \frac{1}{2}$, or of coalition $\{1, 4\}$ if $\alpha_1 < \alpha_2 + \alpha_3 < \frac{1}{2}$, or of coalition $\{2, 3, 4\}$ if $\alpha_2 + \alpha_3 < \alpha_1 < \frac{1}{2}$.

3.1.3 The Case of Five Parties

In the four party case, we have seen that the assumption that small parties are office-seeking is sufficient to guarantee generic existence of the core. On the other hand, it is immediately verified that this assumption also implies that in the four party case the core always consists of the *minimum* winning coalition. Hence, the prediction of our present framework is not different from the previous framework with purely office-seeking parties. However, the picture changes as soon as the number of parties exceeds 4. For instance, with $n = 5$ condition OS applied to the small parties still guarantees generic core existence but the resulting coalitions are not always the minimum winning coalitions. The following result is proved in the appendix.

Fact 3: Suppose that $n = 5$ and the condition (M) holds. Furthermore, suppose that condition OS(3, 4, 5) applies. Then, generically, the core is non-empty and single-valued.

To give an example, where under condition OS(3, 4, 5) the core solution is not the minimum winning coalition, consider the following distribution of weights. Suppose that $(\alpha_1, \ldots, \alpha_5) = (0.28, 0.27, 0.16, 0.15, 0.14)$. In this case, the set of all minimal winning coalitions is

$$\{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\},$$

of which $\{1, 2\}$ is the minimum winning coalition. However, it seems most plausible that party 2 would not opt for $\{1, 2\}$ but for a coalition with two smaller parties since this is the only possibility for 2 to realise its advocated policy program under the plurality rule. Given this preference of party 2 and condition OS(3, 4, 5), the single core coalition is indeed the coalition $\{2, 4, 5\}$. In order to verify this, we first show that $\{2, 4, 5\}$ is undominated among all minimal winning coalitions. By assumption, 2 strictly prefers $\{2, 4, 5\}$ to $\{1, 2\}$. Due to condition (M), 2 strictly prefers $\{2, 4, 5\}$ to both $\{2, 3, 4\}$ and $\{2, 3, 5\}$. By condition OS(4), 4 strictly prefers $\{2, 4, 5\}$ to both $\{1, 3, 4\}$ and $\{1, 4, 5\}$. Similarly, by OS(5), 5 strictly prefers $\{2, 4, 5\}$ to $\{1, 3, 5\}$. This proves that $\{2, 4, 5\}$ is undominated. Next, is is shown that any other minimal winning coalition is dominated. By assumption, $\{1, 2\}$ is dominated by $\{2, 4, 5\}$. Using conditions (M) and OS(3, 4, 5), it is obvious that $\{1, 3, 4\}$ is dominated by $\{1, 3, 5\}$, which in turn is dominated by $\{1, 4, 5\}$. Also, $\{1, 4, 5\}$ is dominated by $\{2, 4, 5\}$, which in turn dominates both $\{2, 3, 4\}$ and $\{2, 3, 5\}$.

3.1.4 A General Core Existence Result

Consider now the case with an arbitrary finite number n of parties. Given our analysis of the cases $n = 3, 4, 5$, it is tempting to conjecture that condition OS, applied to sufficiently many of the small parties, would always guarantee a non-empty core. This, however, is not true as the following example shows.

Example 3. Consider six different parties with weights

$$(\alpha_1, \ldots, \alpha_6) = (0.30, 0.21, 0.14, 0.13, 0.12, 0.10).$$

The set of all minimal winning coalitions is $\{\{1, 2\}\} \cup \mathscr{W}_1 \cup \mathscr{W}_2$, where $\mathscr{W}_1$ denotes the set of all coalitions containing party 1 and exactly two different members of the set $\{3, 4, 5, 6\}$ of all "small" parties, and $\mathscr{W}_2$ denotes the set of all coalitions containing party 2 and exactly three members of $\{3, 4, 5, 6\}$. The minimum winning coalition is $\{1, 2\}$. Assume that all "small" parties (all members of $\{3, 4, 5, 6\}$) are office-seeking, i.e. assume that condition OS (3, 4, 5, 6) holds.[10] Furthermore, assume that despite having less relative power, party 2 strictly prefers the coalitions in $\mathscr{W}_2$ to the minimum winning coalition $\{1, 2\}$. Such a preference seems to be plausible since in any coalition of the set $\mathscr{W}_2$ party 2 can force its own policy program to be realised. In that case, however, the result is that the core is empty. This can be verified as follows. First, observe that any coalition in $\mathscr{W}_1 \setminus \{\{1, 5, 6\}\}$ is dominated by $\{1, 5, 6\}$ due to conditions (M) and OS (3, 4, 5, 6). Similarly, any coalition in $\mathscr{W}_2 \setminus \{\{2, 4, 5, 6\}\}$ is dominated by $\{2, 4, 5, 6\}$. Hence, the only remaining candidates for a possible core solution are the three coalitions $\{1, 2\}$, $\{2, 4, 5, 6\}$ and $\{1, 5, 6\}$. However, our assumptions imply that $\{1, 2\}$ is dominated by $\{2, 4, 5, 6\}$, which in turn is dominated by $\{1, 5, 6\}$. Finally, $\{1, 5, 6\}$ is dominated by $\{1, 2\}$. Consequently, there is no undominated coalition, unless party 2 were willing to sacrifice the realisation of its own advocated policy program for higher relative power.

Example 3 shows that for $n > 5$ stronger conditions than OS are needed in order to insure generic existence of the core. Consider the following strengthening of conditions OS ("party i is strongly office-seeking"):

$$\text{SOS}(i) \qquad \frac{\alpha_i}{\sum_{j\in S}\alpha_j} > \frac{\alpha_i}{\sum_{j\in T}\alpha_j} \Rightarrow \left(x_{i^*_S}, \frac{\alpha_i}{\sum_{j\in S}\alpha_j}\right) \succ_i \left(x_{i^*_T}, \frac{\alpha_i}{\sum_{j\in T}\alpha_j}\right), \text{ for all } S, T \subseteq N.$$

Thus, in contrast to condition OS (i), condition SOS (i) requires party i to be office-seeking even in cases in which this could imply a sacrifie of the realisation of i's advocated policy program. Clearly, SOS (i) is a much stronger condition than OS (i) with less intuitive appeal.[11] In order to formulate the following general result on generic core existence we need the following definition. Sup-

[10] Observe, that we could even require that *all* parties are office-seeking. Indeed, in the above example condition OS is vacuously satisfied for the two big parties 1 and 2. The reason is that there is no winning coalition in which neither 1 nor 2 is the biggest party.

[11] Note, however, that condition SOS would be satisfied if the parties' preferences were *lexicographic* in the way assumed, e.g., in TAYLOR [1972].

pose that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, as is generically the case. The *median party* is the unique party i_M such that i_M is the largest index with $\sum_{j=i_M}^{n} \alpha_j > \frac{1}{2}$. Hence, the median party is the smallest party which may occur in a winning coalition as the biggest member. The following theorem gives sufficient conditions for generic core existence in the general case with arbitrary n.

Theorem 2: Let n be any finite number of parties, and let i_M with $1 < i_M < n$ denote the median party. Suppose that condition (M) holds. Furthermore, assume that condition $\mathrm{OS}(j)$ is satisfied for all $j > i_M$, and that condition $\mathrm{SOS}(j)$ is satisfied for all $j = 2, \ldots, i_M$. Then, generically in the space of all NTU games derived from a distribution $(\alpha_i)_{i \in N}$ of weights according to (3), the core is non-empty and consists of the unique minimum winning coalition.

Proof: In order to prove this, let S be the unique minimum winning coalition and consider any other winning coalition $T \subseteq N$. It will be shown that for some $j \in T$, $w(S)_j \succ_j w(T)_j$. Since both coalitions, S and T, are winning they must have at least one common member $j \in S \cap T$. There are three cases. First, assume that $j > i_M$. Then one obtains $w(S)_j \succ_j w(T)_j$ by condition $\mathrm{OS}(j)$. Indeed, since $j > i_M$, party j can never force its own policy program to be realised, hence j's preference is for S which offers higher relative power. Secondly, assume that $2 \leq j \leq i_M$. In this case, j's strict preference for the minimum winning coalition S stems from condition $\mathrm{SOS}(j)$. Finally, assume that $j = 1$. In this case, party 1 can realise its advocated policy program in S as well as in T. Hence, due to condition (M), party 1 strictly prefers the minimum winning coalition S. This shows that the minimum winning coalition is indeed undominated, hence an element of the core.

On the other hand, suppose that T is a winning coalition but not the minimum winning coalition S. It will be shown that S dominates T. First, let j be the biggest party in S, i.e. $j = i_S^*$. Then, since S has less total weight, j strictly prefers coalition S over T due to condition (M). Next, suppose that $j \in S$ is not the biggest party in S. If j is the biggest party in T, it follows that $j \in \{2, \ldots, i_M\}$. Consequently, j's strict preference for S over T is implied by condition $\mathrm{SOS}(j)$. If on the other hand $j \in S$ is not the biggest party in T, then the strict preference $w(S)_j \succ_j w(T)_j$ is implied by condition $\mathrm{OS}(j)$ (recall that $\mathrm{SOS}(j) \Rightarrow \mathrm{OS}(j)$, so that $\mathrm{OS}(j)$ is satisfied for all $j = 2, \ldots, n$). Summarising, for all $j \in S$, $w(S)_j \succ_j w(T)_j$, hence T is dominated by S. *Q.E.D.*

It should be emphasised that theorem 2 does not go too much beyond what has already been established in theorem 1. The main difference between the two results is that the assumption of strongly office-seeking behaviour which is implicit in theorem 1 for *all* parties, can by theorem 2 be restricted to parties $2, \ldots, i_M$ if policy is determined by plurality voting within winning coalitions. On the other hand, as example 3 shows one cannot completely dispense with the assumption of strongly office-seeking behaviour. Hence, it is at least doubtful whether one can obtain stronger general results than theorem 2 in the scenario with plurality voting.

3.2 Policy Compatibility

In this subsection, we offer an alternative approach to the problem of policy bargaining within winning coalitions. The basic idea is that for each party the desirability of a coalition depends on the degree of *compatibility* of the policy programs advocated by the different coalition members. Specifically, it is assumed that coalitions with parties whose policy programs are "ideologically closer" to each other are preferred. Indeed, these are the coalitions that are more likely to function properly. In order to provide a rigorous formal account of this idea, we first introduce a measure of ideological "closeness" of different political parties. Hence, for all $i, j \in N$, let $d(i,j) \in \mathbb{R}$ denote the *distance* of party i from party j in terms of political preferences. It will be assumed that the function $d: N \times N \to [0, \infty)$ satisfies the following two properties. For all i, j, $d(i,j) = 0 \Leftrightarrow i = j$, and $d(i,j) = d(j,i)$. In the following analysis, no other properties of the distance function d are used, hence we do not have to specify a particular form of this function. Note, however, that in any particular application the distance function would have to be derived from the parties' preferences over different policy programs. Hence, in the present framework too, policy preferences explicitly enter the analysis.

Given a distance function $d: N \times N \to [0, \infty)$, consider the following specific payoff function $u: \mathscr{W} \to \mathbb{R}^N$. Given any winning coalition $S \in \mathscr{W}$, the payoff ("utility") for each $i \in N$ is given by

$$(4) \qquad u(S)_i = \begin{cases} g\left(\sum_{j \in S} \frac{\alpha_j}{\sum_{k \in S} \alpha_k} d(i,j) \right) & \text{if } i \in S \\ 0 & \text{if } i \notin S, \end{cases}$$

where $g: \mathbb{R}_+ \to \mathbb{R}_+$ is any strictly decreasing function. Hence, the utility of each coalition member is strictly decreasing in the weighted sum of the distances to each other coalition member, where the weights are given by the respective relative power within the coalition. In particular, given $i \in S$, i's utility of S is the lower the more distant and bigger parties coalition S contains.

It is emphasised that a specification of the payoff function such as (4) avoids the problem of determining the outcome of policy bargaining within coalitions. Indeed, (4) entails no explicit specification of the actual outcome of a bargaining process. All that is assumed is that for each coalition partner the outcome will be more favourable if there are less distant and relatively powerful parties. In its spirit, the postulation of a payoff function such as (4) is close to the approaches of AXELROD [1970] and DE SWAAN [1973]. However, unlike these authors the present model does not assume a one-dimensional policy space.[12] Indeed, the applicability of the "right–left" distinction, and the assumption of one single policy dimension in general, has become even more doubtful since the rise of the Green parties in Europe.

We note that the specification (4) of the payoff function implies that parties which are not member of a winning coalition are indifferent as between policy outcomes. This may be a drawback but seems to be unavoidable in our present model.

3.2.1 The Case of Three Parties

As in our previous framework, we first consider the simplest non-trivial case of three parties. Interestingly, in this case the core is generically non-empty and single-valued for *any* given distance function. Formally, one has the following result.

Theorem 3: Let $n = 3$, and let d be any given metric on the set $\{1, 2, 3\}$. Generically in the space of all three person NTU games derived from a distribution $(\alpha_i)_{i=1,2,3}$ according to (4), the core is non-empty and single-valued.

Proof: The proof is given by a contradiction argument. First, it is easily verified that, generically, no party is indifferent between two distinct winning coalitions. Observe that this immediately implies in the three party case that the core must be single-valued if it exists. Hence, if suffices to verify existence of the core. It is also clear that the grand coalition is dominated by any two-party coalition (recall that we assume $\alpha_1 < \frac{1}{2}$, hence any two-party coalition is winning). Now assume that the core is empty, and consider party 1. Generically, either 1 strictly prefers to form a coalition with party 2, or it strictly prefers a coalition with party 3. In the following, we only consider the first case since the other case is completely symmetric. In the first case, one has

$$(5) \qquad \frac{\alpha_2}{\alpha_1 + \alpha_2} d(1, 2) < \frac{\alpha_3}{\alpha_1 + \alpha_3} d(1, 3) .$$

Since we suppose that the core is empty, it must be the case that party 2 on the other hand strictly prefers to form a coalition with party 3 (otherwise $\{1, 2\}$

[12] Axelrod's "minimal connected winning" theory assumes that parties can be ideologically ordered according to one single policy dimension (e.g. by means of the "right–left" distinction) and predicts that only ideologically *connected* coalitions will form. Here, "ideologically connected" means that all members of the coalition are adjacent to each other on the single policy dimension. De Swaan generalises this approach using a cardinal measure on the single policy dimension and develops a "closed minimal range" theory of coalition formation.

would be a core solution). Consequently, one must have

$$\frac{\alpha_3}{\alpha_2 + \alpha_3} d(2, 3) < \frac{\alpha_1}{\alpha_1 + \alpha_2} d(1, 2) . \tag{6}$$

By the same argument, party 3 strictly prefers to form a coalition with party 1, hence

$$\frac{\alpha_1}{\alpha_1 + \alpha_3} d(1, 3) < \frac{\alpha_2}{\alpha_2 + \alpha_3} d(2, 3) . \tag{7}$$

From (5) one obtains

$$\frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{\alpha_1 + \alpha_3}{\alpha_3} d(1, 2) < d(1, 3) , \tag{8}$$

and from (6) one obtains

$$d(2, 3) < \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{\alpha_2 + \alpha_3}{\alpha_3} d(1, 2) . \tag{9}$$

Substituting (8) and (9) in (7) yields

$$\frac{\alpha_1}{\alpha_1 + \alpha_3} \cdot \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{\alpha_1 + \alpha_3}{\alpha_3} d(1, 2) < \frac{\alpha_2}{\alpha_2 + \alpha_3} \cdot \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{\alpha_2 + \alpha_3}{\alpha_3} d(1, 2),$$

which reduces to

$$\frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)\alpha_3} < \frac{\alpha_2 \alpha_1}{(\alpha_1 + \alpha_2)\alpha_3},$$

an obvious contradiction. Thus, the core cannot be empty. *Q.E.D.*

It is emphasised that our present model allows for big parties' coalitions. Indeed, typically a coalition between the two big parties will occur whenever the distance $d(1, 2)$ is sufficiently smaller than either $d(1, 3)$ and $d(2, 3)$. As an example consider the following scenario. Suppose that the distribution of weights is (0.42, 0.40, 0.18), and that the distances are given by $d(1, 2) = 1$ and $d(1, 3) = d(2, 3) = 2$. Also, let g be given by $g(z) = 1/z$. In this case, the payoffs for parties 1 and 2 are

$$u(\{1, 2\})_1 = 2.05 \quad \text{and} \quad u(\{1, 3\})_1 = 1.667 ,$$
$$u(\{1, 2\})_2 = 1.952 \quad \text{and} \quad u(\{2, 3\})_2 = 1.611 ,$$

hence the big parties' coalition is the unique core solution.

3.2.2 A General Result

Without further restrictions, the core will typically be empty when more than three parties are involved. In this paragraph, we informally state a sufficient condition for generic core existence with an arbitrary number of parties. The condition is based on the observation that with a constant distance between any pair of parties all that matters according to the payoff function (4) is again each party's relative power within a coalition. Indeed, suppose that for all distinct i, j and all distinct k, l one has $d(i, j) = d(k, l)$. Then it is easily verified that the payoff function (4) is ordinally equivalent to the payoff function (1) defined in section 2. Consequently, by theorem 1 the core generically exists and contains only the minimum winning coalition. Since generically all parties' preferences between different winning coalitions are strict, the same conclusion must apply if the distance function is sufficiently close to a constant function. Hence, one obtains the following result.

Fact 4: Generically in the space of all NTU games derived from some distribution of weights $(\alpha_i)_{i \in N}$ according to (4), the core is non-empty and consists of the unique minimum winning coalition if the distance function $d: N \times N \rightarrow [0, \infty)$ is sufficiently close to some constant function.

Appendix: Generic Core Existence in the Five Party Case Under Plurality Voting

In this appendix, we derive generic core existence with five parties in the framework of subsection 3.1. Hence, assume that the five parties are endowed with weights $\frac{1}{2} > \alpha_1 > \alpha_2 > \ldots > \alpha_5 > 0$, as is generically the case. Consider the NTU game induced by (3), and suppose that conditions (M) and OS(3, 4, 5) are satisfied. The following case distinction is based on which two-party coalitions are winning. It is easily verified that the cases listed below are mutually exclusive and exhaustive.

Case 1: No coalition of two parties is winning, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.24, 0.23, 0.19, 0.18, 0.16).$$

In this case, the set of all minimal winning coalitions is exactly the set of all three party coalitions. The minimum winning coalition is $\{3, 4, 5\}$. Clearly, party 3 strictly prefers this coalition to any other coalition due to condition (M). Since 4 and 5 are assumed to be office-seeking, the coalition $\{3, 4, 5\}$ is also strictly preferred by 4 and 5, hence it is the unique core solution.

Case 2: $\{1, 2\}$ is winning, but not $\{1, 3\}$. In this case, the set of all minimal winning coalitions is

$$\{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\}.$$

Case 2a: $\alpha_1 > \alpha_4 + \alpha_5$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.37, 0.34, 0.11, 0.10, 0.08).$$

In this case, $\{2, 4, 5\}$ is the minimum winning coalition. Since parties 4 and 5 are assumed to be office seeking, it is therefore the unique core solution.

Case 2b: $\alpha_1 < \alpha_4 + \alpha_5$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.28, 0.27, 0.16, 0.15, 0.14).$$

In this case, $\{1, 2\}$ is the minimum winning coalition. It is easily verified that party 2 is the *pivotal party*. Indeed, if party 2's preference is

$$\left(x_2, \frac{\alpha_2}{\alpha_2 + \alpha_4 + \alpha_5}\right) \succ_2 \left(x_1, \frac{\alpha_2}{\alpha_1 + \alpha_2}\right),$$

as may most plausibly be assumed (see subsection 3.1.3), the unique core solution is $\{2, 4, 5\}$. On the other hand, if party 2 happens to be strongly office-seeking, i.e. if 2's preference is

$$\left(x_2, \frac{\alpha_2}{\alpha_2 + \alpha_4 + \alpha_5}\right) \prec_2 \left(x_1, \frac{\alpha_2}{\alpha_1 + \alpha_2}\right),$$

then $\{1, 2\}$ is the unique core coalition.

Case 3: $\{1, 3\}$ is winning, but not $\{1, 4\}$.

Case 3a: In addition, $\{2, 3\}$ is winning, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.33, 0.30, 0.27, 0.06, 0.04).$$

In this case, the set of all minimal winning coalitions consists of $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. Clearly, the minimum winning coalition is $\{2, 3\}$, and, since party 3 is assumed to be office-seeking, it is indeed the unique core solution.

Case 3b: In addition, $\{2, 3\}$ is not winning. In this case, the set of all minimal winning coalitions is

$$\{\{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}.$$

Again, we have to distinguish several cases:

(i) $\alpha_3 < \alpha_4 + \alpha_5$ and $\alpha_2 + \alpha_5 < \alpha_1$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.32, 0.27, 0.20, 0.17, 0.04) .$$

It is easily verified that the assumptions of this case imply that $\{2, 3, 5\}$ is the minimum winning coalition. Therefore, since parties 3 and 5 are assumed to be office-seeking, $\{2, 3, 5\}$ is the unique core solution.

(ii) $\alpha_3 < \alpha_4 + \alpha_5$ and $\alpha_2 + \alpha_5 > \alpha_1$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.32, 0.28, 0.19, 0.16, 0.05) .$$

Under this assumption, $\{1, 3\}$ is the minimum winning coalition and the unique core solution due to condition OS(3).

(iii) $\alpha_3 > \alpha_4 + \alpha_5$ and $\alpha_2 + \alpha_3 < \alpha_1 + \alpha_4$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.34, 0.27, 0.20, 0.15, 0.04) .$$

Now, the minimum winning coalition is $\{2, 3, 5\}$. It is also the unique core solution, by means of condition OS(3, 5).

(iv) $\alpha_3 > \alpha_4 + \alpha_5$ and $\alpha_2 + \alpha_3 > \alpha_1 + \alpha_4$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.32, 0.29, 0.20, 0.15, 0.04) .$$

The minimum winning coalition in this case is $\{1, 4, 5\}$, and because of condition OS(4, 5) it is also the unique core solution.

Case 4:. $\{1, 4\}$ is winning, but not $\{1, 5\}$. In this case, the set of all minimal winning coalition is

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\} .$$

Case 4a: $\alpha_1 < \alpha_2 + \alpha_3$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.35, 0.21, 0.20, 0.19, 0.05) .$$

Since party 4 is office-seeking, the minimum winning coalition $\{1, 4\}$ is the unique core solution.

Case 4b: $\alpha_1 > \alpha_2 + \alpha_3$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.40, 0.20, 0.18, 0.17, 0.05) .$$

In this case, the minimum winning coalition is $\{2, 3, 4\}$. It is also the unique core solution due to condition OS(3, 4).

Case 5: $\{1, 5\}$ is winning. The minimal winning coalitions are

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3, 4, 5\}\} .$$

Case 5a: $\alpha_1 > \alpha_2 + \alpha_3 + \alpha_4$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.48, 0.16, 0.13, 0.12, 0.11) .$$

The minimum winning coalition in this case is $\{2, 3, 4, 5\}$, and due to condition OS(3, 4, 5) it is the unique core solution.

Case 5b: $\alpha_1 < \alpha_2 + \alpha_3 + \alpha_4$, e.g.

$$(\alpha_1, \ldots, \alpha_5) = (0.38, 0.17, 0.16, 0.15, 0.14) .$$

In this final case, the minimum winning coalition is $\{1, 5\}$. Conditions OS(5) implies that it is the unique core solution.

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