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INTRANSITIVE CHOICES BASED ON TRANSITIVE PREFERENCES: THE CASE OF MENU-DEPENDENT INFORMATION

ABSTRACT. The paper considers decision contexts in which the set of alternatives from which choices have to be made (the 'menu') may convey information about the desirability of these alternatives. Our analysis is motivated by the fact that in specifying an appropriate description of a choice situation an outside observer always has to neglect some 'dimensions' of the decision problem. The central claim is that properties of observed behaviour may depend crucially on the neglected dimensions even when their influence is *arbitrarily small*. Specifically, we prove the following result. Suppose that in a discrete choice model an agent's beliefs about the 'quality' of the available alternatives depend on the specific menu to choose from. Then, even when the difference in beliefs given different menus is arbitrarily small (but positive), any practicable description of the decision situation necessarily implies that *cyclic choices* will be observed. The result suggests that transitivity – as a condition on observable behaviour – is rather questionable in any context where one cannot *completely* exclude the possibility that menu-dependent information may play some role.

KEY WORDS: Transitivity of preferences, intransitive choice behaviour, menu-dependent information, choice set specifications.

1. INTRODUCTION

One of the cornerstones of preference modelling in economics is the assumption that preferences over choice objects are transitive. The standard text-book justification for this assumption is the claim that any conceivable notion of a decision maker's rationality would entail transitivity of her[1] preferences. Despite this, there is a persistent interest in models of choice behaviour which accommodate intransitive preferences (see Fishburn, 1982, 1984a, 1984b; Bell, 1982; Loomes and Sugden, 1982, among others). Indeed, a number of authors have suggested that there are decision contexts in which intransitive choices not only frequently arise but seem to follow well-defined and consistent guidelines (see e.g. May, 1954; Tversky, 1969; Grether and Plott 1979; and for more recent contributions

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Bar-Hillel and Margalit, 1988; Fishburn and LaValle, 1988; Anand, 1993a, 1993b, among others).

The present paper focuses on one such context, in which the set of alternatives (the 'menu') from which choices have to be made may convey information about the desirability of the alternatives. Our central claim is that in such contexts any plausible and practicable description of the decision situation will *imply* intransitive choices. To make our point clear, we do not merely claim that one can construct specific examples in which intransitive behaviour is likely to be observed in reality. Rather, we will demonstrate the following. Suppose that in a class of decision problems the different menus from which choices have to be made induce different beliefs about the quality of the available alternatives. Then, *any* 'rational' decision maker will exhibit cyclic choice behaviour under some configuration of beliefs. Moreover, this will be true even when the difference in beliefs given different menus is arbitrarily small but positive. Hence, the only case in which intransitive choice behaviour may never occur is the case in which information is *independent* of the specific menu at hand.

It is emphasized that this does not entail the claim that the *preferences* of these decision makers are necessarily intransitive. This would follow only under additional assumptions, such as e.g. the hypothesis of classical revealed preference theory that preference is nothing but what is revealed by choices. However, this hypothesis – although very familiar in economics – is far from being uncontroversial (see e.g. Sen, 1973, among many other critical discussions of the conceptual difficulties of the classical approach of revealed preference theory). Indeed, it has been argued that what is revealed by choices may crucially depend on the *external context* in which the choices are performed (see, e.g. Sen, 1993; Anand, 1993b). The present paper models such a context explicitly. As we shall see, this leads to conclusions very different from the standard paradigm of rational behaviour.

We do not want to advocate any theory about the conceptual relation between 'choice' and 'preference' here. However, we will have to discuss a different conceptual issue concerning the question of what has to be regarded the appropriate *description* of a given decision context. It has been argued that in many cases violations of transitivity could be removed by redefining the choice primitives, thus by changing the description of the choice problem. While this might be true, it seems that this argument raises more problems than it solves. Indeed, Anand (1990) has shown that as any 'intransitive' behaviour can be given a transitive description, so too can any 'transitive' behaviour be given an intransitive description (see also Anand, 1993b). Thus, for practical purposes the question is whether for a given choice situation there are descriptions which are more *plausible* and more *practicable* than others. We will argue that in the choice contexts considered in this paper, the descriptions which would correspond to transitive behaviour are not only less practicable but are *prima facie* also less plausible. The reason is that the descriptions corresponding to transitive behaviour require information which, in general, will not be available to an outside observer *before* choices are performed.

To illustrate the point, consider the following example which is a variant of a well-known example due to Luce and Raiffa (1957).

Example. A lady wandering in a strange city at dinner time chances upon a modest restaurant which she enters uncertainly. The waiter informs her that there is no menu, but that this evening she may have either broiled salmon at $2.50 (remember, all this takes place in 1957) or steak at $4.00. In a first-rate restaurant her choice would have been steak, but considering her unknown surroundings and the different prices she elects the salmon. Next evening, our lady enters a different restaurant where the waiter informs her that this evening there are frog legs at $4.50 and broiled salmon at $2.50 on the menu. It so happens that our lady really likes frog legs, although she would always prefer a well prepared steak in a restaurant as good as she assumes this one to be. Since steak is not available that night she chooses frog legs. On the final evening of her stay in that strange city our lady enters a third restaurant where there is a small but obviously selected menu consisting of frog legs at $4.50 and steak at $4.00. 'Splendid, she thinks, I'll have the steak tonight'.

Obviously, the choices described in the example are intransitive. But are these choices irrational? The intended interpretation of the above example is as follows. Suppose that our lady in that strange city has the following (transitive!) preferences. She strictly prefers a high-quality steak to either frog legs and salmon. Further, she strictly prefers frog legs to salmon, and either to a low-quality steak. Now, suppose that our lady takes the presence of frog legs on the menu as an indicator of the high quality of the restaurant. Then her choices would be exactly as described in the example. Clearly, these choices admit a transitive description. Indeed, we just gave one such description.[2] So shouldn't we rather describe the above example in terms of high and low quality steaks, rendering the lady's choices not only completely rational but also transitive? The problem with this suggestion is how an outside observer could detect that the steaks at the different restaurants are in fact different alternatives *for the decision maker*. Clearly, one could examine the steaks, finding that the steak in the first restaurant was of low quality whereas the steak in the last restaurant met all standards one could expect in a first-rate restaurant. That doesn't solve the problem since what is relevant to choice is not the true quality of the steak but the decision maker's *beliefs* about that quality. It could well have turned out that the two steaks were indistinguishable with respect to their quality. Still, choice behaviour wouldn't be different given a decision maker's wrong beliefs.

Typically, an outside observer, or an experimenter, has to specify the universal set of alternatives *before* he can observe choice behaviour. Thus, a certain description of the choice objects must be given in advance. Would it be plausible to distinguish different qualities in the above example *before* the observed choices have occurred? Suppose that the steaks in the above example were physically indistinguishable and that choice behaviour was guided by the decision maker's wrong beliefs about the quality of the steaks. In this case, it is not clear how different qualities could enter the description of the decision situation specified by an outside observer. Indeed, it seems obvious enough that *prima facie* any plausible description of the choice objects must be based on the information available to the outside observer. Given this, the outside observer must treat the two steaks in the above example as two instances of the same alternative. But then the above choices *are* intransitive.

Again, one might argue that *after* the choices in the above example were observed a description involving different qualities and resulting in transitive behaviour would be more plausible. However, it seems that this argument is just based on the intuition that transitive behaviour is more plausible than intransitive behaviour. But if transitivity is a property of behaviour only *relative* to a certain description of the choice situation, it follows that "transitivity is a matter of language, not of behaviour" (Anand, 1993b, p. 103). In particular, if transitivity is the point at issue, then the plausibility of a certain description must not be judged by whether or not it yields transitive behaviour.

Besides the conceptual problems in determining the most plausible description of a given choice situation, there is also a more practical problem which may play an important role in our present context. In specifying the precise nature of a choice situation an outside observer always has to neglect at least some aspects, or 'dimensions', of the decision problem. Indeed, in order to obtain a practicable description of a choice situation one has to focus on those features which seem to be most relevant for the decision. Clearly, one can never be sure that the neglected 'dimensions' are indeed irrelevant for the decision. But one might hope that the procedure is justified as soon as the influence of the neglected 'dimensions' is small. However, the results to be presented here demonstrate that this is, in general, not the case. Although the present paper focuses only on one particular context, where the neglected 'dimension' is given by the possibility of menu-dependent information, the general problem concerning neglected 'dimensions' presumably arises in various other decision contexts as well.

The subsequent sections are organized as follows. Section 2 introduces our basic choice model which incorporates the possibility of menu-dependent information. In Section 3 it is shown that in this model intransitive choices necessarily occur even when the difference in beliefs given different menus becomes arbitrarily small. The main argument relies on a certain version of Brouwer's Fixed Point Theorem. It is noted that the result is completely general in the sense that it applies to virtually *any* choice function specifying how a decision maker's choices relate to her preferences and her beliefs. In Section 4 it is shown that, under very mild additional assumptions, the set of beliefs which induce intransitive behaviour has *full dimension* in the belief space. Hence, intransitive choices are not the exception. Concluding remarks are given in Section 5.

2. A DISCRETE CHOICE MODEL WITH MENU-DEPENDENT INFORMATION

Let X with $\#X \geqslant 3$ be a finite set of choice objects which are available in different unobservable quality. Let $x, y, z \in X$ be three distinct elements of X. Assume that due to the different qualities in which these alternatives are available a decision maker has the following possible *transitive* preferences among those three alternatives.

$$
\begin{aligned}
\succ_1 &: x \succ y, y \succ z \quad \text{and} \quad x \succ z \\
\succ_2 &: x \succ z, z \succ y \quad \text{and} \quad x \succ y \\
\succ_3 &: y \succ z, z \succ x \quad \text{and} \quad y \succ x \\
\succ_4 &: y \succ x, x \succ z \quad \text{and} \quad y \succ z \\
\succ_5 &: z \succ x, x \succ y \quad \text{and} \quad z \succ y \\
\succ_6 &: z \succ y, y \succ x \quad \text{and} \quad z \succ x.
\end{aligned}
$$

The intended interpretation is that these preference orderings correspond to different combinations of the qualities of the alternatives. Thus, for instance, if x is of high quality but y and z are of low quality, the decision maker's preferences would be as in $\succ_1$. If, on the other hand, x and z were of high quality and y of low quality, then her preferences would be according to $\succ_2$, and so on. Note that our decision maker is a completely rational person with transitive preferences in each case.[3] For simplicity we rule out indifferences between x, y and z. We assume that the decision maker has beliefs about the quality of the alternatives. Our central assumption is that the belief about the quality of a certain alternative may depend on the specific menu in which it occurs. Thus, as in the example in the introduction, we allow for the possibility that the presence of other alternatives in a menu may serve as an indicator for the quality of a specific alternative. Given our six possible preference orderings the beliefs about quality can be summarized in a belief about which of the six preference orderings is relevant to the choice from a given menu.[4] Specifically, consider the three menus $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$. Clearly, the decision context considered here may involve other menus and other alternatives as well. However, for our purposes it will suffice to consider the three alternatives $x, y, z \in X$ and the three menus above. For $i = 1, \ldots, 6$, denote by $q_i^{x,y}$ the subjective probability that, given the menu $\{x, y\}$ to choose from, the qualities of x, y, z are such that the ordering $\succ_i$ applies. Let S^5 denote the five-dimensional simplex in $\mathbf{R}^5_+$, i.e. $S^5 := \{(q_1, \ldots, q_5) \in \mathbf{R}^5_+ : \sum_{i=1}^5 q_i \leqslant 1\}$. Note that any element $(q_1, \ldots, q_5) \in S^5$ uniquely determines the last coordinate, q_6, via $q_6 = 1 - \sum_{i=1}^5 q_i$. The decision maker's beliefs given the menu $\{x, y\}$ can thus be described by an element $q^{x,y} = (q_1^{x,y}, \ldots, q_5^{x,y})$ of S^5. In the context of the three menus above, the decision maker's overall beliefs can be summarized in a vector $\hat{q} \in S^5 \times S^5 \times S^5$, where $\hat{q} = (q^{x,y}, q^{x,z}, q^{y,z})$.[5]

Next, we have to specify how a decision maker's choices relate to her beliefs. In our formal analysis, we do not want to restrict ourselves to specific forms of this relation. Thus we do not want to propose any specific theory of choice under uncertainty here. In contrast, our analysis will apply to a very large class of possible choice functions. We will, however, make the general assumption that given a certain menu there is a fixed deterministic relationship between a decision maker's beliefs and her choices. Consider a specific menu, say $\{x, y\}$. Denote by $\mathcal{P}_{x,y}$ the set $2^{\{x,y\}} \setminus \{\emptyset\}$, i.e. the set of all non-empty subsets of $\{x, y\}$. Let $C^{x,y} : S^5 \to \mathcal{P}_{x,y}$ denote the choice function specifying for each $q \in S^5$ the resulting choice from menu $\{x, y\}$. Note that $C^{x,y}$ is not assumed to be single-valued. The sets $\mathcal{P}_{y,z}, \mathcal{P}_{x,z}$, and the functions $C^{y,z} : S^5 \to \mathcal{P}_{y,z}, C^{x,z} : S^5 \to \mathcal{P}_{x,z}$ are similarly defined. The three choice functions are summarized in a function

$$C : S^5 \times S^5 \times S^5 \to \mathcal{P}_{x,y} \times \mathcal{P}_{y,z} \times \mathcal{P}_{x,z},$$

where $C(\hat{q}) = (C^{x,y}(q^{x,y}), C^{y,z}(q^{y,z}), C^{x,z}(q^{x,z}))$. Note that our description implicitly assumes that given a specific menu, say $\{x, y\}$, a decision maker's choice depends only on $q^{x,y}$, i.e. on her beliefs *given* $\{x, y\}$, and not on her beliefs in the counterfactual situations where she would have to choose from $\{y, z\}$, or from $\{x, z\}$.

It is clear that without further specification, a choice rule C can accommodate almost every kind of behaviour, be it rational or irrational. In order to exclude overtly unreasonable choices, we use the following definition. Denote by $q^{x,y}(x \succ y)$ the probability that, given the menu $\{x, y\}$, the alternative x will be strictly preferred to y. Hence,

$$q^{x,y}(x \succ y) = q_1^{x,y} + q_2^{x,y} + q_5^{x,y}.$$

The numbers $q^{y,z}(y \succ z)$ and $q^{x,z}(x \succ z)$ are similarly defined.

DEFINITION 1. The choice rule $C^{x,y} : S^5 \to \mathcal{P}_{x,y}$ is called *weakly admissible* if and only if $C^{x,y}(q^{x,y}) = \{x\}$ whenever $q^{x,y}(x \succ y) = 1$, and $C^{x,y}(q^{x,y}) = \{y\}$ whenever $q^{x,y}(x \succ y) = 0$.

Thus, $C^{x,y}$ is weakly admissible if and only if it always picks x from $\{x, y\}$ when the decision maker is sure to strictly prefer x to y, and it always picks y from $\{x, y\}$ when the decision maker is sure to strictly prefer y to x (given the menu $\{x, y\}$). Similarly, we define weak admissibility for the choice rules $C^{y,z}$ and $C^{x,z}$. Finally, say that $C = (C^{x,y}, C^{y,z}, C^{x,z})$ is *weakly admissible* if and only if $C^{x,y}, C^{y,z}$, and $C^{x,z}$ are weakly admissible. We do not claim that every weakly admissible choice rule C corresponds to rational behaviour. However, it seems that, conversely, any rational choice rule must be weakly admissible.

One possible specification for the choice rule C is of course the expected utility form. Indeed, as described in Note 4, one could interpret alternatives as lotteries over different qualities and require the axioms of expected utility to hold for preferences over these lotteries. Note that maximizing expected utility in such a framework would yield a weakly admissible choice rule. However, the formulation of the expected utility choice rule would require explicit reference to an appropriate state space, thereby unnecessarily complicating the model. Moreover, it seems doubtful whether the axioms of expected utility have great appeal in our specific context. Therefore, we do not restrict our analysis to the expected utility form but consider a much larger class of choice rules in which maximizing expected utility is just a special case.

To conclude the description of our model we turn to the central issue of menu-dependent information. Say that a certain decision context, i.e. a set of possible menus, is *information-neutral* if and only if a decision maker's beliefs about the quality of the alternatives do not depend on the specific menu offered to her. Denote by Δ the diagonal in $S^5 \times S^5 \times S^5$, i.e.

$$\Delta := \{(q, q', q'') \in S^5 \times S^5 \times S^5 : q = q' = q''\}.$$

Thus, our present decision context given by the three menus $\{x, y\}$, $\{y, z\}$, and $\{x, z\}$ would be information-neutral if and only if $\hat{q} \in \Delta$, i.e. if and only if $q^{x,y} = q^{y,z} = q^{x,z}$. If on the other hand, the probabilities $q^{x,y}$, $q^{y,z}$ and $q^{x,z}$ are distinct, the decision context properly involves menu-dependent information in the sense that the decision maker's beliefs are different given different menus to choose from. It is this case in which we are primarily interested. Intuitively, the Euclidean distance of $\hat{q} \in S^5 \times S^5 \times S^5$ to the diagonal Δ measures the extent of menu-dependent information. Indeed, $\hat{q}$ is 'near' the diagonal if and only if the distance between $q^{x,y}, q^{y,z}$ and $q^{x,z}$ is small, i.e. if and only if the difference in the decision maker's beliefs given each of the three menus $\{x, y\}$, $\{y, z\}$ and $\{x, z\}$ is small. For $\epsilon \geqslant 0$, let Δ_ϵ denote the set of all $\hat{q}$ whose Euclidean distance to Δ is less than, or equal to ϵ, i.e.

$$\Delta_\epsilon := \bigcup_{\hat{q} \in \Delta} \{\hat{q}' \in S^5 \times S^5 \times S^5 : d(\hat{q}', \hat{q}) \leqslant \epsilon\},$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance. In particular, $\Delta_0 = \Delta$.[6]

3. THE EXISTENCE OF BELIEFS INDUCING CYCLIC CHOICES

In our definition of a choice rule we have allowed for the possibility that there are ties between two alternatives. In this section, we will assume that ties are broken according to some fixed *deterministic* tie breaking rule. This amounts to assuming the choice function to be single-valued. Under this assumption we will prove the following theorem.

THEOREM 1. *Let C be a weakly admissible, single-valued choice rule. Then, for any $\epsilon > 0$, there exists $\hat{q} \in \Delta_\epsilon$ such that $C(\hat{q}) = (\{x\}, \{y\}, \{z\})$.*

Note that the conclusion of Theorem 1 implies intransitive choices. Indeed, suppose that a decision maker's beliefs are represented by $\hat{q} \in S^5 \times S^5 \times S^5$. Then $C(\hat{q}) = (\{x\}, \{y\}, \{z\})$ just means that the decision maker chooses x from $\{x, y\}$, y from $\{y, z\}$, and z from $\{x, z\}$, hence 'revealing' the preference cycle $x \succ y \succ z \succ x$. Thus, Theorem 1 shows that for any weakly admissible, single-valued choice rule there exist beliefs which necessarily induce intransitive behaviour. Furthermore, these beliefs can be chosen arbitrarily close to the diagonal, i.e. the differences in beliefs given the different menus can be chosen arbitrarily small. In contrast, the claim of Theorem 1 would be false for $\epsilon = 0$, i.e. in the case where the decision maker considers the decision context to be information-neutral. This can be seen by means of the following example.

Example. Define a choice rule as follows. Let $i_0(x, y)$ denote the minimal index such that $q^{x,y}_{i_0(x,y)} \geqslant q^{x,y}_i$ for all $i = 1, \ldots, 6$. Then, given the menu $\{x, y\}$ choose according to the preference order $\succ_{i_0(x,y)}$. Hence, choices are made according to the preference order which has highest probability given the menu at hand. If there are ties in the highest probability, choose the preference order with lowest index. Similarly, define the choice rule given the other two menus. If the decision context is information-neutral, then clearly $i_0(x, y) = i_0(y, z) = i_0(x, z)$, so that the same transitive preference order applies to each of the three menus. Consequently, in that case the choice rule always results in transitive choices. Note that the choice rule just described is weakly admissible and single-valued. In particular, by Theorem 1 this choice rule also induces intransitive behaviour for some beliefs arbitrarily close to the neutral case.

Proof of Theorem 1. The proof is given in two steps. First, the problem of finding $\hat{q} \in \Delta_\epsilon$ as required in Theorem 1 is reduced to the problem of finding a suitable point in a two-dimensional subset of S^5. The existence of such a point is then guaranteed using an appropriate version of Brouwer's Fixed Point Theorem (see Hurewicz and Wallman, 1948, pp. 40–41).

Consider in S^5 the four points $e_6 := (0, 0, 0, 0, 0)$, $e_1 := (1, 0, 0, 0, 0)$, $e_2 := (0, 1, 0, 0, 0)$, and $e_3 := (0, 0, 1, 0, 0)$. Let

$$\begin{aligned} V_1 &:= \{(\lambda, 1 - \lambda, 0, 0, 0) : \lambda \in [0, 1]\}, \\ V_2 &:= \{(0, 0, \lambda, 0, 0) : \lambda \in [0, 1]\}, \\ W_1 &:= \{(\lambda, 0, 1 - \lambda, 0, 0) : \lambda \in [0, 1]\} \\ W_2 &:= \{(0, \lambda, 0, 0, 0) : \lambda \in [0, 1]\}. \end{aligned}$$

Hence, V_1 is the edge of S^5 connecting e_1 and e_2, and V_2 is the edge connecting e_6 and e_3. Similarly, W_1 is the edge connecting e_1 and e_3, and W_2 connects e_6 and e_2. (See Figure 1, which shows the three-dimensional projection of S^5 onto the first three coordinates. Note

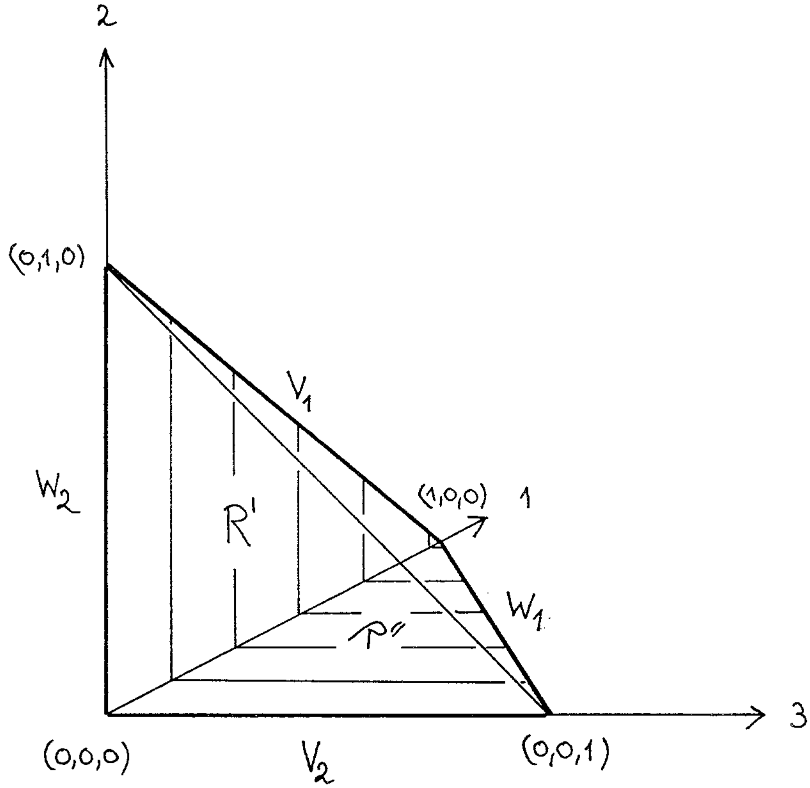

Fig. 1. Three-dimensional projection of S^5 onto the first three coordinates.

that in the figure, axis i refers to the weight of preference ordering $\succ_i$ for $i = 1, 2, 3$.)

The sets $V_i, i = 1, 2$, are chosen in such a way that on V_1 the alternative x is with certainty strictly preferred to the alternative y. Conversely, on V_2 the alternative y is with certainty strictly preferred to the alternative x. Indeed, any point of V_1 only gives weight to the preference orderings $\succ_1$ and $\succ_2$, both of which rank x strictly above y. On the other hand, any point of V_2 only gives weight to $\succ_3$ and $\succ_6$, both of which rank y strictly above x. Similarly, y is with certainty strictly preferred to z on W_1, and z is with certainty strictly preferred to y on W_2.

Denote by R' the surface of S^5 spanned by V_1 and W_2, and by R'' the surface spanned by V_2 and W_1. Finally, let $R := R' \cup R''$. Next, we define

$$A_1 := \{q \in R : C^{x,y}(q) = \{x\}\},$$

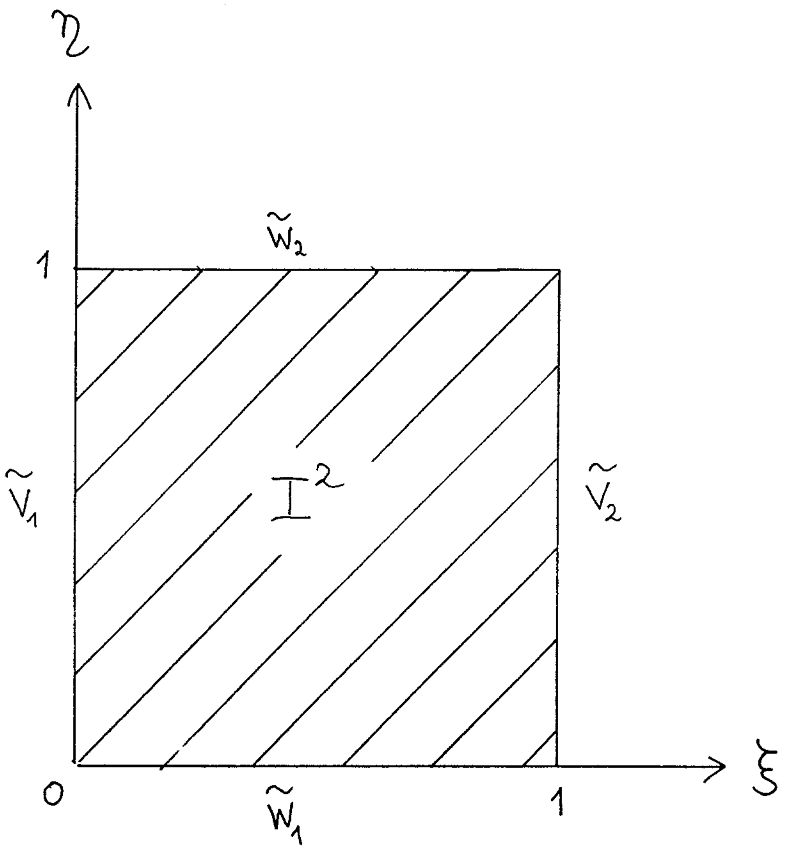

Fig. 2. Unit square in $\mathbf{R}^2$.

$$
\begin{aligned}
A_2 &:= \{q \in R : C^{x,y}(q) = \{y\}\},\\
B_1 &:= \{q \in R : C^{y,z}(q) = \{y\}\},\\
B_2 &:= \{q \in R : C^{y,z}(q) = \{z\}\}.
\end{aligned}
$$

By weak admissibility of the choice rule C, one has $V_i \subseteq A_i$ and $W_i \subseteq B_i$ for $i = 1, 2$. In particular, the sets A_i and B_i, $i = 1, 2$, are non-empty. Hence, by single-valuedness, both $\{A_1, A_2\}$ and $\{B_1, B_2\}$ form a partition of R.

The set $R \subseteq S^5$ is homeomorphic to the unit square in $\mathbf{R}^2_+$, which we denote by I^2. Accordingly, choose a homeomorphism $f : R \to I^2$ which maps V_1 and V_2, as well as W_1 and W_2, into opposite faces of the unit square, respectively. Thus, denoting by $\tilde{V}_i$ and $\tilde{W}_i$ the images of V_i and W_i, respectively, the picture looks as shown in Figure 2.

For $i = 1, 2$, let $\tilde{A}_i := f(A_i)$ and $\tilde{B}_i := f(B_i)$. Clearly, $\tilde{V}_i \subseteq \tilde{A}_i$ and $\tilde{W}_i \subseteq \tilde{B}_i$ for $i = 1, 2$. Denote by $bd\tilde{A}_1$ and $bd\tilde{B}_1$ the boundary of $\tilde{A}_1$ and $\tilde{B}_1$ in I^2, respectively. Note that $bd\tilde{A}_1 = bd\tilde{A}_2$ and $bd\tilde{B}_1 = bd\tilde{B}_2$.

Define a mapping $g : I^2 \to I^2$ as follows.

$$g(\xi, \eta) := (\xi \pm d((\xi, \eta), bd\tilde{A}_1), \eta \pm d((\xi, \eta), bd\tilde{B}_1)),$$

where d denotes the Euclidean distance in $\mathbf{R}^2$ and the sign in the definition of g being determined as follows. If $(\xi, \eta) \in \tilde{A}_1$ the sign in the first coordinate is taken as $+$, if on the other hand $(\xi, \eta) \in \tilde{A}_2$ the sign in the first coordinate is taken as $-$. Similarly, the sign in the second coordinate is taken as $+$ if $(\xi, \eta) \in \tilde{B}_1$, and as $-$ if $(\xi, \eta) \in \tilde{B}_2$. It is easily verified that g maps I^2 into itself and is continuous. Hence, by Brouwer's Theorem there exists a fixed point (ξ^0, η^0) of g. By the definition of g,

$$d((\xi^0, \eta^0), bd\tilde{A}_1) = d((\xi^0, \eta^0), bd\tilde{B}_1) = 0.$$

Since $bd\tilde{A}_1$ and $bd\tilde{B}_1$ are closed this implies $(\xi^0, \eta^0) \in bd\tilde{A}_1 \cap bd\tilde{B}_1$.

Consider the inverse image $q^0 \in R$ of (ξ^0, η^0) under f. Since f is a homeomorphism one has $q^0 \in bdA_1 \cap bdB_1$. Therefore, since both $\{A_1, A_2\}$ and $\{B_1, B_2\}$ form a partition of R, in every neighbourhood of q^0 there exist $q^1, q^2, \bar{q}^1, \bar{q}^2$ such that $q^i \in A_i$ and $\bar{q}^i \in B_i$ for $i = 1$, 2, i.e.

$$\begin{aligned} C^{x,y}(q^1) &= \{x\}, \\ C^{x,y}(q^2) &= \{y\}, \\ C^{y,z}(\bar{q}^1) &= \{y\}, \\ C^{y,z}(\bar{q}^2) &= \{z\}. \end{aligned}$$

Finally, consider the choice function $C^{x,z}$. First, assume that $C^{x,z}(q^0) = \{z\}$. Then for any given $\epsilon > 0$ one finds q^1 and $\bar{q}^1$ as above such that $\hat{q} := (q^1, \bar{q}^1, q^0) \in \Delta_\epsilon$ and $C(\hat{q}) = (\{x\}, \{y\}, \{z\})$. If, on the other hand, $C^{x,z}(q^0) = \{x\}$ choose q^2 and $\bar{q}^2$ as above such that x is chosen from $\{x, z\}$, z is chosen from $\{z, y\}$, and y is chosen from $\{y, x\}$. However, which of the two cycles occurs is only a matter of the labelling of the alternatives. Hence, by suitably relabelling the alternatives one can always obtain $C(\hat{q}) = (\{x\}, \{y\}, \{z\})$ with $\hat{q}$ arbitrarily close to Δ. This completes the proof of Theorem 1.

4. ON THE SIZE OF THE SET OF BELIEFS INDUCING CYCLIC CHOICES

Having established the existence of beliefs which induce intransitive choice behaviour the question arises whether such choice behaviour is likely to be observed. In this section we will prove that under very mild additional assumptions there is indeed a *positive probability* that intransitive choices will be observed. Specifically, it is proved that the set of beliefs which induce cyclic choices has positive Lebesgue-measure in the belief space. First, it is noted that weak admissibility of the choice rule is not sufficient to imply this.

Example. Define a choice rule $C = (C^{x,y}, C^{y,z}, C^{x,z})$ as follows.

$$\begin{aligned} C^{x,y}(q^{x,y}) &= \begin{cases} \{y\} & \text{if } q^{x,y}(x \succ y) = 0 \\ \{x\} & \text{if } q^{x,y}(x \succ y) > 0, \end{cases} \\ C^{y,z}(q^{y,z}) &= \begin{cases} \{z\} & \text{if } q^{y,z}(y \succ z) = 0 \\ \{y\} & \text{if } q^{y,z}(y \succ z) > 0, \end{cases} \\ C^{x,z}(q^{x,z}) &= \begin{cases} \{z\} & \text{if } q^{x,z}(x \succ z) = 0 \\ \{x\} & \text{if } q^{x,z}(x \succ z) > 0. \end{cases} \end{aligned}$$

Hence, x is always chosen from $\{x, y\}$ except in the case where $q^{x,y}(x \succ y) = 0$. Similarly, y is always chosen from $\{y, z\}$ except when $q^{y,z}(y \succ z) = 0$, and x is always chosen from $\{x, z\}$ except when $q^{x,z}(x \succ z) = 0$. It is easily verified that given this choice function cyclic choices can only occur for a belief $\hat{q} = (q^{x,y}, q^{y,z}, q^{x,z})$ such that at least one of the probabilities $q^{x,y}$, $q^{y,z}$, or $q^{x,z}$ is an element of the boundary of S^5. Indeed, if all three probabilities are in the interior of S^5 the choice rule prescribes the choice of x from $\{x, y\}, y$ from $\{y, z\}$, and again x from $\{x, z\}$. Obviously, these choices do not form a cycle. Consequently, the set of beliefs for which the above rule induces cyclic choices must have measure zero (in $S^5 \times S^5 \times S^5$).

Although the choice rule considered in the example is weakly admissible in the sense of Definition 1, the choice behaviour it prescribes does not seem to be very reasonable. For example, suppose that $q^{x,y}(x \succ y)$ is very small but positive, i.e. the subjective probability that the qualities of x and y (given the menu $\{x, y\}$ to choose from) are such that x is preferred to y is very small. It seems that y is the only rational choice in this case. Nevertheless, the choice rule in the above example picks x instead. In order to exclude such behaviour, we introduce the following slightly stronger notion of admissibility.

DEFINITION 2. A choice rule $C^{x,y} : S^5 \to \mathcal{P}_{x,y}$ is called *admissible* if and only if there exists a positive real number δ such that $C^{x,y}(q^{x,y}) = \{x\}$ whenever $q^{x,y}(x \succ y) > 1 - \delta$, and $C^{x,y}(q^{x,y}) = \{y\}$ whenever $q^{x,y}(x \succ y) < \delta$.

In contrast to weak admissibility in the sense of Definition 1, admissibility in the sense of Definition 2 requires that x is always chosen from $\{x, y\}$ if the decision maker is *almost* sure to strictly prefer x to y, and similarly, that y is always chosen from $\{x, y\}$ if y is almost surely preferred to x.

Admissibility for the choice rules $C^{y,z}$, $C^{x,z}$, and for the overall choice rule C is defined in obvious analogy. It is emphasized that even this stronger notion of admissibility is certainly not sufficient to guarantee 'rationality' of the choice rule. But again, it seems to be a necessary condition for any notion of rational behaviour in the present context. Note that admissibility in the sense of Definition 2 could be derived from weak admissibility in the sense of Definition 1 by a simple *continuity* argument.

A further necessary condition for a choice rule to induce 'rational' behaviour would be some sort of monotonicity property. Suppose that for some belief $q^{x,y}$ one has $x \in C^{x,y}(q^{x,y})$. Now assume that the probability that x is preferred to y (given $\{x, y\}$) rises. Then, it seems that the only rational thing to do is to choose x from $\{x, y\}$. The following definition captures this idea.

DEFINITION 3. The choice rule $C^{x,y} : S^5 \to \mathcal{P}_{x,y}$ is called *locally monotone* if and only if for all $q^{x,y}, \bar{q}^{x,y} \in S^5$ such that

$$\begin{aligned} \bar{q}_i^{x,y} &> q_i^{x,y} \quad \text{for } i = 1, 2, 5, \\ \bar{q}_i^{x,y} &< q_i^{x,y} \quad \text{for } i = 3, 4, \text{ and} \end{aligned}$$

$$\sum_{i=1}^{5} \bar{q}_i^{x,y} > \sum_{i=1}^{5} q_i^{x,y},$$

one has

$$x \in C^{x,y}(q^{x,y}) \Rightarrow C^{x,y}(\bar{q}^{x,y}) = \{x\}, \quad \text{and}$$
$$y \in C^{x,y}(\bar{q}^{x,y}) \Rightarrow C^{x,y}(q^{x,y}) = \{y\}.$$

Note that the conditions on $q^{x,y}$ and $\bar{q}^{x,y}$ in Definition 3 just state that compared with $q^{x,y}$, the belief $\bar{q}^{x,y}$ gives more weight to each of the preference orderings which favour x over y, and less weight to each ordering which favours y over x. Hence, the choice function $C^{x,y}$ is locally monotone if and only if x is always chosen at a certain belief given that x was already in the choice set at a belief giving less weight to each preference ordering favouring x and more weight to each preference ordering favouring y. Similarly, y is always chosen given that y was already in the choice set at a belief giving less weight to each ordering favouring y and more weight to each ordering favouring x. We note that in the model described in Note 4 the property of local monotonicity would be implied by the requirement that preferences respect the principle of *stochastic dominance*.

The definition of local monotonicity applies in an analogous way to the choice functions $C^{y,z}$ and $C^{x,z}$. Note, however, that the conditions by which the two beliefs q and $\bar{q}$ are compared apply to different coordinates in each case. Finally, the overall choice rule $C = (C^{x,y}, C^{y,z}C^{x,z})$ is called locally monotone if every coordinate is locally monotone.

THEOREM 2. *Let C be an admissible, locally monotone choice rule. Denote by $\mathcal{I} \subseteq S^5 \times S^5 \times S^5$ the set of beliefs which induce cyclic choices. Then for every $\epsilon > 0$, the set $\mathcal{I} \cap \Delta_\epsilon$ contains an open subset of $S^5 \times S^5 \times S^5$.*

Clearly, if $\mathcal{I} \cap \Delta_\epsilon$ contains an open subset it must have positive measure in $S^5 \times S^5 \times S^5$. Note that Theorem 2 does not require single-valuedness of C. In particular, it also applies to the case where ties in the choice function are broken according to some *probabilistic* tie breaking rule.

Proof. The proof uses essentially the same geometric argument as the proof of Theorem 1. Let δ be a fixed positive real number with $0 < \delta < 1/5$. Consider in S^5 the following four points.

$$P_0 := (\delta, \delta, \delta, \delta, \delta),$$

$$\begin{aligned} P_1 &:= (1-5\delta, \delta, \delta, \delta, \delta), \\ P_2 &:= (\delta, 1-5\delta, \delta, \delta, \delta), \quad \text{and} \\ P_3 &:= (\delta, \delta, 1-5\delta, \delta, \delta). \end{aligned}$$

Define the sets V_i and W_i, $i = 1, 2$, as follows. V_1 is the line segment connecting P_1 and P_2, and V_2 is the line segment connecting P_0 and P_3. Furthermore, W_1 is the line segment connecting P_1 and P_3, and W_2 is the line segment connecting P_0 and P_2. Denote by R' the plane segment spanned by V_1 and W_2, and by R'' the plane segment spanned by V_2 and W_1. Thus, the picture looks just as shown in Figure 1 except that the surfaces are slightly shifted into the interior of S^5. Finally, let R denote the union of R' and R'', and set

$$\begin{aligned} A_1 &:= \{q \in R : x \in C^{x,y}(q)\}, \\ A_2 &:= \{q \in R : x \notin C^{x,y}(q)\}, \\ B_1 &:= \{q \in R : y \in C^{y,z}(q)\}, \\ B_2 &:= \{q \in R : y \notin C^{y,z}(q)\}. \end{aligned}$$

Obviously, both $\{A_1, A_2\}$ and $\{B_1, B_2\}$ form a partition of the set R. It is easily verified that $q \in V_1$ implies $q(x \succ y) = 1 - 3\delta$, and $q \in V_2$ implies $q(x \succ y) = 3\delta$. Similarly, $q \in W_1$ implies $q(y \succ z) = 1 - 3\delta$, and $q \in W_2$ implies $q(y \succ z) = 3\delta$. Hence, by admissibility of the choice rule, if δ is sufficiently small one obtains $V_i \subseteq A_i$ and $W_i \subseteq B_i$ for $i = 1, 2$.

As in the proof of Theorem 1 one can now apply Brouwer's Fixed Point Theorem in order to establish the existence of q^0 such that $q^0 \in bdA_1 \cap bdB_1$. By construction, q^0 must be a point of the interior of S^5. Furthermore, by local monotonicity one can find in every neighborhood of q^0 points $q^1, q^2, \bar{q}^1, \bar{q}^2$ such that

$$\begin{aligned} C^{x,y}(q^1) &= \{x\}, \\ C^{x,y}(q^2) &= \{y\}, \\ C^{y,z}(\bar{q}^1) &= \{y\}, \\ C^{y,z}(\bar{q}^2) &= \{z\}. \end{aligned}$$

This can be seen as follows. Since q^0 is an element of bdA_1 there must exist in every neighborhood of q^0 an element p such that $x \in C^{x,y}(p)$. Now suppose that q^1 gives slightly more weight than p to each preference ordering favouring x over y and slightly less weight to each preference ordering favouring y over x. Then, $C^{x,y}(q^1) = \{x\}$ by local monotonicity. The existence of $q^2, \bar{q}^1$, and $\bar{q}^2$ as required above is shown by a similar argument.

Again by local monotonicity, in every neighbourhood of q^1 one can even find an open subset U^1 such that

$$C^{x,y}(q) = \{x\} \quad \text{for all } q \in U^1.$$

Indeed, suppose that some belief p in a neighbourhood of q^1 gives more weight than q^1 to each preference ordering favouring x over y and less weight to each preference ordering favouring y over x. Then this will be true even in a small neighbourhood U^1 of p. Hence, by local monotonicity $C^{x,y}(q) = \{x\}$ must hold in this neighbourhood. Similarly, in every neighbourhood of $q^2, \bar{q}^1$, and $\bar{q}^2$ there exist open subsets $U^2, \bar{U}^1$, and $\bar{U}^2$, respectively, such that

$$\begin{aligned} C^{x,y}(q) &= \{y\} \quad \text{for all } q \in U^2, \\ C^{y,z}(q) &= \{y\} \quad \text{for all } q \in \bar{U}^1, \\ C^{y,z}(q) &= \{z\} \quad \text{for all } q \in \bar{U}^2. \end{aligned}$$

Given the point q^0 from above, there are two possible cases. Either, (i) $z \in C^{x,z}(q^0)$, or (ii) $z \notin C^{x,z}(q^0)$.

Case (i): Given any $\epsilon > 0$, choose $\tilde{q}^1$ sufficiently close to q^0 such that $C^{x,z}(\tilde{q}^1) = \{z\}$. Furthermore, let $\tilde{U}^1$ be an open set sufficiently close to $\tilde{q}^1$ such that $C^{x,z}(q) = \{z\}$ for all $q \in \tilde{U}^1$. The existence of $\tilde{q}^1$ and $\tilde{U}^1$ is again guaranteed by local monotonicity. By construction, the set $U^1 \times \bar{U}^1 \times \tilde{U}^1$ is open in $S^5 \times S^5 \times S^5$, and for any $\hat{q} \in U^1 \times \bar{U}^1 \times \tilde{U}^1$ one obtains the choice cycle $C(\hat{q}) = (\{x\}, \{y\}, \{z\})$. Hence, if U^1 and $\bar{U}^1$ are sufficiently close to q^1 and $\bar{q}^1$, respectively, one has

$$U^1 \times \bar{U}^1 \times \tilde{U}^1 \subseteq \mathcal{I} \cap \Delta_\epsilon.$$

Case (ii). The argument in the second case is completely symmetric. Indeed, as in Case (i) one can find open sets $U^2, \bar{U}^2$ and $\tilde{U}^2$ such that for any $\hat{q} \in U^2 \times \bar{U}^2 \times \tilde{U}^2$ the choice cycle $C(\hat{q}) = (\{y\}, \{z\}, \{x\})$ results. By choosing $U^2, \bar{U}^2$ and $\tilde{U}^2$ sufficiently close to $q^2, \bar{q}^2$ and $\tilde{q}^2$, respectively, one finally obtains

$$U^2 \times \bar{U}^2 \times \tilde{U}^2 \subseteq \mathcal{I} \cap \Delta_\epsilon.$$

This completes the proof of Theorem 2.

5. CONCLUSION

Over the past years, several authors have questioned the assumption of transitive choice behaviour based on the observation that in many circumstances preferences of rational agents might be intransitive. In this paper, we have tried to argue that the case for transitive *behaviour* might be even worse. In a model of menu-dependent information it has been shown that even when *preferences* are transitive, cyclic behaviour necessarily occurs as soon as the choice alternatives are not fully specified in all relevant 'dimensions'. Hence, transitivity as a property of behaviour seems to be particularly sensitive to the precise description of the choice alternatives. This result can be of some practical importance, since in many cases it may be hard to attain the full specification needed for transitive behaviour.

In our approach, the central issue of menu-dependence has been modelled using varying probabilities on the space of the relevant preference orderings. As a different approach, one might try to incorporate menu-dependence using the concept of valued preference relations on the set of alternatives (for models of valued preference relations see e.g. Ovchinnikov 1981; Roubens and Vincke 1987; or Barrett and Pattanaik 1989).[7] Indeed, in the context of valued preferences menu-dependent choices may quite naturally arise (e.g. some of the choice rules considered in Barrett, Pattanaik and Salles, 1990, induce menu-dependent choices). The concept of valued preferences may also be used to explain cyclic choices such as the choice of x from $\{x, y\}$, y from $\{y, z\}$ and z from $\{x, z\}$. However, the central result of this paper is that such choice behaviour may arise even when the choice mechanism is entirely based on *transitive* preferences. Hence, in order to reproduce such a result in terms of valued preferences one would have to have a clear concept of transitivity for valued relations. But the question of the appropriate definition of transitivity for valued relations is controversially discussed in the literature, and consequently different competing notions of transitivity have been suggested (see e.g. Barrett and Pattanaik, 1989). Furthermore, it can be shown that – under reasonable additional assumptions – some of the notions of transitivity suggested in the literature do allow for the cyclic behaviour described above whereas other notions of transitivity do not allow for such behaviour.[8] Besides these difficulties with the appropriate definition of transitivity the concept of valued preference relations might be fruitfully applied to menu-dependent choices. This, however, is beyond the scope of the present paper and needs separate investigation.

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NOTES

[1] The sex of the generic decision maker referred to in this paper has been determined by chance (tossing a fair coin). It turned out to be female.

[2] There are, of course, other descriptions which would not imply intransitive choices. For instance, one could assume non-separable preferences over time, treating the meals at different dates as different alternatives. This would not solve our issue since we could slightly change the example assuming the choices to be hypothetical choices to be performed at the same time.

[3] This feature distinguishes our approach from the approach of Fishburn and LaValle (1988) who analyse context-dependent choice in a different framework where they allow for intransitive *preferences*.

[4] In the following, the beliefs about the relevant preference orderings given a certain menu are taken to be the primitives of the choice model. Clearly, it would be possible to derive these beliefs from a prior on an appropriate state space. For instance, one could explicitly introduce different quality levels for each alternative, treating the alternatives as *lotteries* over the different quality levels. Due to the presence of menu-dependent information, the prior probability for the quality levels would depend on the specific menu at hand. Each combination of quality levels would correspond to a certain preference ordering $\succ_i$. The belief about the relevant preference ordering given a certain menu to choose from would then result from updating the prior probability over the quality levels given that menu.

[5] Throughout, we will use the 'hat' symbol to denote elements of $S^5 \times S^5 \times S^5$. Thus, generic elements of $S^5 \times S^5 \times S^5$ are denoted by $\hat{q}$, whereas generic elements of S^5 are denoted by q.

[6] Although the intended interpretation of our model is in terms of *individual* decision making, there might also be an interesting interpretation in a social choice context. Consider a fixed population of individuals each of whom possesses one of the transitive preference orderings $\succ_1$ through $\succ_6$. Suppose that for each different menu there is a different subset of individuals who are faced with a choice from that menu. Then, for instance, $q_i^{x,y}$ could be interpreted as the fraction of

those individuals who are faced with a choice from menu $\{x, y\}$ and whose preferences are described by the ordering $\succ_i$. In this framework, our condition of weak admissibility would readily translate into the familiar unanimity principle. Furthermore, what we have called the information neutral case would correspond to the case in which the fraction q_i of individuals with preference ordering $\succ_i$ is independent from the specific menu under consideration. We are indebted to an anonymous referee for suggesting this interpretation.

[7] We are grateful to an anonymous referee for this hint.

[8] Let $R : X \times X \to [0, 1]$ be a valued binary relation on X. Assume that for any $w, v \in X$, the choice set from $\{w, v\}$ is equal to $\{w\}$ if and only if $R(w, v) > R(v, w)$. One can easily show the following two statements. (i) Suppose that R satisfies Max-Min Transitivity (see e.g. Barrett and Patanaik, 1989). Furthermore, assume that x is the only alternative chosen from $\{x, y\}$, and y is the only alternative chosen from $\{y, z\}$. Then, x must be chosen from $\{x, z\}$. (ii) There exist specifications of R such that R is connected and satisfies Sum-Minus-One Transitivity (see Barrett and Pattanaik, 1989), and such that the choice sets from $\{x, y\}, \{y, z\}$, and $\{x, z\}$ are $\{x\}, \{y\}$, and $\{z\}$, respectively.

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