Lecture 8. Repeated Games

What happens if the same game is repeated? Can promises and threats about the future change the behavior?

1. Finitely repeated games

Definition: Given a static game G, the finitely repeated game $G(T)$ is a game in which G is played T times. For each $t \leq T$, the outcomes of the $t-1$ preceding repetitions is known to the players before repetition t begins. Denote by π_i^t the payoff of player i from repetition t. The overall payoffs of player of the whole game is the sum of his payoffs of the T static games, i.e.

$$\pi_i^1 + \pi_i^2 + ...\pi_i^T = \sum_{t=1}^{T} \pi_i^t$$

To distinguish between the whole repeated game, G is also called "stage game".

Example: cooperation game (prisoner's dilemma), repeated twice.

	C_2		NC_2	
C_1	2	2	0	3
NC_1	3	0	1	1

This repeated game is a two-stage game with imperfect information (last lecture) $\Rightarrow$ *SPE*

Second stage game: Only *NE* of the second stage is (NC_1, NC_2), independently of what has happened in the first stage game.

First stage game: If in the second stage players play (NC_1, NC_2) independently of the result of first stage, (NC_1, NC_2) is the only equilibrium in the first stage, too.

$\Rightarrow$ *SPE* of the twice repeated cooperation game is just twice the *NE* of the stage game.

Proposition: If the stage game G has a unique NE, then for any finite T the repeated game $G(T)$ has a unique subgame-perfect outcome: the NE of G is played in every stage.

This proposition does not hold if the stage game has more than one NE.

Example: stage game G, repeated twice

	L		M		R	
U	1	1	5	0	0	0
C	0	5	4	4	0	0
B	0	0	0	0	3	3

Note: The stage game G has two NE in pure strategies: (U, L) and (B, R)

Take the following strategies of the players for the whole repeated game $G(2)$

row player:

1. Stage (Repetition): C
2. Stage (Repetition): $\begin{cases} B & \text{if 1.Stage resulted in } (C, M) \\ U & \text{if 1.Stage did not result in } (C, M) \end{cases}$

column player

1. Stage (Repetition) : M
2. Stage (Repetition) : $\begin{cases} R & \text{if 1.Stage resulted in } (C, M) \\ L & \text{if 1.Stage did not result in } (C, M) \end{cases}$

Claim: This stategy combination is a SPE.

Proof: i) We have to show that for any outcome of the first stage the second stage behavior is consistent with a *SPE*.
Take the first stage outcome (C, M) with payoffs $(4, 4)$. For this first stage outcome and the connected payoffs, the game and its payoffs become

	L		M		R	
U	$1+4$	$1+4$	$5+4$	$0+4$	$0+4$	$0+4$
C	$0+4$	$5+4$	$4+4$	$4+4$	$0+4$	$0+4$
B	$0+4$	$0+4$	$0+4$	$0+4$	$3+4$	$3+4$

For this static game, (B, R) is a *NE*.

Take any first stage outcome that is not (C, M), and denote the connected payoffs by (x, y). For this first stage outcome and the connected payoffs, the game and its payoffs become

	L		M		R	
U	$1+x$	$1+y$	$5+x$	$0+y$	$0+x$	$0+y$
C	$0+x$	$5+y$	$4+x$	$4+y$	$0+x$	$0+y$
B	$0+x$	$0+y$	$0+x$	$0+y$	$3+x$	$3+y$

For any (x, y), (U, L) is a NE.

$\Rightarrow$ second stage behavior is consistent with SPE - in fact any behavior that is an NE of the stage game G

ii) We have to show that for the proposed second stage actions, the first stage actions form a *NE* of the whole game.
Taken the second stage actions as decribed above into account, the payoffs are given by:

	L		M		R	
U	$1+1$	$1+1$	$5+1$	$0+1$	$0+1$	$0+1$
C	$0+1$	$5+1$	$4+3$	$4+3$	$0+1$	$0+1$
B	$0+1$	$0+1$	$0+1$	$0+1$	$3+1$	$3+1$

Note: For first stage outcomes (C, M) the second stage payoffs are $(3, 3)$, whereas for all other first stage outcomes the second stage payoffs are $(1, 1) \Rightarrow$

(C, M) in the first stage is part of an *SPE*.■

The possibility to make the seond stage behavior dependent on the outcome of the first stage allows for "non-NE" behavior in the first stage. But this possibility only exists when the second stage behavior is not uniquely determined, i.e. G has more than one *NE*.

2. Infinitely repeated games

Definition: Given a static game, G, $G(\infty, \delta)$ denotes the infinitely repeated game in which G is repeated forever. For each t the outcomes of the $t-1$ preceding repetitions is known to the players before repetition t begins. Denote by π_i^t the payoff of player i from repetition t. The overall payoffs of player of the whole game is the sum of his payoffs of all static games discounted by δ, i.e.

$$\pi_i^1 + \delta\pi_i^2 + \delta^2\pi_i^3 + ... = \sum_{t=1}^{\infty} \delta^{t-1}\pi_i^t$$

Alternatively, one can also use the average discounted payoff per stage

$$(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi_i^t$$

Note that without discounting nothing prevents the payoff to go to ∞.

Folk-theorem: Let $(\pi_1, ...\pi_n)$ be the payoff from a NE of the stage game G, and let $(x_1, ...x_n)$ be any other feasible payoff vector of the stage game G with $x_i > \pi_i$ for all $i = 1, ..n$. If δ is sufficiently close to one, then there exists a SPE of the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, ...x_n)$ as average payoff.

"In an infinitely repeated game anything that is individually rational can be supported by a SPE."

Example: Cooperation game

	C_2		NC_2	
C_1	2	2	0	3
NC_1	3	0	1	1