Lecture 7. Simple Dynamic Games

1. Two-Stage Games of Complete and Perfect Information

Two-Stages dynamic game with two players:

player 1 chooses action a_1 from the set of his feasible actions A_1

player 2 observes a_1 and chooses then action a_2 from the set of feasible actions $A_2(a_1)$. Note: A_2 might depend on a_1.

payoffs: $u_1(a_1, a_2), u_2(a_1, a_2)$

Solved by Backward Induction:

Step 1: determine optimal action of 2 for given a_1.

$$\max_{a_2 \in A_2(a_1)} u_2(a_1, a_2)$$

Solution to the maximization problem: $BR_2(a_1)$. Note: for each a_1 there might be a different optimal action of 2.

Assume that $BR_2(a_1)$ is unique for all a_1

Step 2: player 1 foresees the optimal response of 2 to any a_1. Knowing $2's$ reaction, player 1 chooses his optimal action.

$$\max_{a_1 \in A_1} u_1(a_1, BR_2(a_1))$$

solution determines backward induction outcome (a_1^*, a_2^*) with $a_2^* = BR_2(a_1^*)$.

Example: Stackelberg game

Same market as in Cournot duopoly game

homogenous good

linear market demand Q at price P:

$$Q = \left\{ \begin{array}{cc} 1 - P & \text{if } P \leq 1 \\ 0 & \text{if } P > 1 \end{array} \right\}$$

or

$$P = \left\{ \begin{array}{cc} 1 - Q & \text{if } Q \leq 1 \\ 0 & \text{if } Q > 1 \end{array} \right\}$$

Good produced by 2 firms with no fix cost and constant marginal costs c. $c < 1$.

Stage 1: Firm 1 ("Stackelberg leader") chooses its quantity q_1.

Stage 2: After observing $1's$ quantity, 2 chooses its quantity q_2.

Difference to Cournot: In Cournot game, both firms choose quantities at same time - firm 2 cannot make its quantitity choice dependent on q_1.

payoff-functions

$$\begin{aligned} \pi_i(q_i, q_j) &= q_i.P - q_i.c \\ &= \left\{ \begin{array}{cc} q_i(1-(q_i+q_j)) - q_i.c & \text{if } q_i + q_j \leq 1 \\ -q_i.c & \text{if } q_i + q_j > 1 \end{array} \right\} \end{aligned}$$

Backward induction

What is optimal for 2 if 1 chooses q_1?

$$\underset{q_2 \in \mathbb{R}_+}{Max}\ \pi_2(q_2, q_1)$$

$$\frac{\partial \pi_2(q_2, q_1)}{\partial q_2} = 1 - 2q_2 - q_1 - c \Longrightarrow$$

If $1 - q_1 - c \geq 0 : BR_2(q_1) = \frac{1-q_1-c}{2}$

If $1 - q_1 - c < 0 : BR_2(q_1) = 0$

Reaction function:

$$BR_2(q_1) = \left\{ \begin{array}{cc} \frac{1-q_1-c}{2} & \text{if } 1-q_1-c \geq 0 \\ 0 & \text{if } 1-q_1-c < 0 \end{array} \right\}$$

Optimal choice of 1:

$$\underset{q_1 \in \mathbb{R}_+}{Max}\ \pi_1(q_1, BR_2(q_1))$$

If $1's$ optimal choice q_1^* would be such that $1 - q_1^* - c < 0$, implying that $P < c$, and therefore $q_1^* = 0$, leading to $1 - q_1^* - c > 0$, which is a contradiction.

So q_1^* must be such that $1 - q_1^* - c \geq 0$ holds, implying that $BR_2(q_1) = \frac{1-q_1-c}{2}$.

$$\underset{q_1 \in \mathbb{R}_+}{Max}\, q_1(1-(q_1+\frac{1-q_1-c}{2}))-q_1.c$$

$$\Rightarrow \quad 0 = 1 - 2q_1^* - \frac{1}{2} + \frac{c}{2} + q_1^* - c$$

Backward Induction outcome:

$$q_1^* = \frac{1-c}{2}$$

$$q_2^* = BR_2(q_1^*) = \frac{1-c}{4}$$

Difference to Cournot: Leader produces a higher, follower a lower quantity.

Reason for difference: In Cournot, firm $1's$ choice of q_1 has no direct impact on $2's$ choice, since both choices are made simultaneously. Within the Stackelberg game, firm $1's$ choice of q_1 induces a certain reaction of 2, namely $BR_2(q_1)$. Rational player 1 takes this reaction into acount.

2. Multiple-Stage Games of Complete and Perfect Information

Backward induction can be used for games with more than two stages

Example: Sequential Bargaining

How to share a certain amount of money, say 1 euro, among two players?

Stage 1: Player 1 proposes that he gets share s_1 of the euro, and player 2 the rest.

Stage 2: Player 2 can accept or recject. If she accepts, the stage 1 proposal s_1 is implemented. Otherwise, the game continues with stage 3.

Stage 3: Player 2 proposes that player 1 gets share s_2, and she gets the rest.

Stage 4: If player 1 accepts, the proposal is implemented; otherwise player 1 receives his outside option o_1 and player 2 his outside option o_2, with $o_1 + o_2 < 1$.

For every 2 stages of delay of agreement, the depreciation factor is $\delta < 1$.

Backward induction

Stage 4: Player 1 accepts offer s_2, if acceptance gives him higher utility than rejecting. Hence, 1 accepts whenever $s_2 \geq o_1$.

Stage 3: Player 2 can either make an offer acceptable to player 1 or not. If she makes an unacceptable offer, she gets o_2. If she makes an acceptable offer, 2 is best off if she makes the lowest offer just acceptable to player 1, i.e. $s_2 = o_1$. In this case 2 receives $1 - o_1$. Since $1 - o_1 > o_2$, 2 will indeed make the lowest acceptable offer, i.e. $s_2 = o_1$.

Stage 2: Player 2 accepts offer s_1, if acceptance gives her higher utility than going to stage 3. If stage 3 is reached, player 2 gets $1 - o_1$, which is worth - viewed from stage 2 - $(1 - o_1)\delta$. Hence, 2 accepts whenever $1 - s_1 \geq (1 - o_1)\delta$.

Stage 1: Player 1 can either make an offer acceptable to player 2 or not. If he makes an unacceptable offer, stage 3 is reached. In this case 1 gets o_1, which is worth - viewed from Stage 1 - δo_1. If 1 makes an acceptable offer, 1 is best off if he makes the lowest offer just acceptable to player 2, i.e. $1 - s_1 = (1 - o_1)\delta$. In this case 1 receives $s_1 = 1 - \delta + \delta o_1$. Since $1 - \delta + \delta o_1 > \delta o_1$, 1 will indeed make the lowest acceptable offer, i.e $s_1 = 1 - \delta + \delta o_1$.

Backward induction equilibrium

$$\begin{aligned}
\text{Stage 1} &: \quad s_1^* = 1 - \delta + \delta o_1 \\
\text{Stage 2} &: \quad \text{Player 2 accepts every offer } s_1 \text{ with } 1 - s_1 \geq (1 - o_1)\delta \\
\text{Stage 3} &: \quad s_2^* = o_1 \\
\text{Stage 4} &: \quad \text{Player 1 accepts every offer } s_2 \text{ with } s_2 \geq o_1
\end{aligned}$$

3. Two-stage dynamic game of complete but imperfect information

imperfect information: Players make simultaneous moves

Two-stage games of imperfect information with 2 players at each stage

Stage 1: Players 1 and 2 choose simultaneously actions a_1 and a_2 from the sets of feasible actions A_1 and A_2.

Stage 2: Players 3 and 4 observe a_1 and a_2. Then they choose simultaneously actions a_3 and a_4 from the sets of feasible actions $A_3(a_1, a_2)$ and $A_4(a_1, a_2)$.

payoffs: $u_i(a_1, a_2, a_3, a_4), i = 1...4$

Player 1 and 3, or player 2 and 4 might be the same (but not 1 and 2, or 3 and 4)

Because of simultaneous choices, backward induction does not work.

$\Rightarrow$ Subgame-perfect equilibrium (SPE)

For fixed (a_1, a_2), players 3 and 4 play a static (simultaneous move) game with strategy sets $A_3(a_1, a_2)$ and $A_4(a_1, a_2)$,and payoffs $u_i(a_1, a_2, a_3, a_4)$, $i = 3, 4$. Assume for the moment that for each (a_1, a_2) the static game has a unique NE in pure strategies, denoted by $(a_3^*(a_1, a_2),\ a_4^*(a_1, a_2))$.

Given that players 3 and 4 play the NE $(a_3^*(a_1, a_2), a_4^*(a_1, a_2))$ for each (a_1, a_2), players 1 and 2 play a static (simultaneous move) game with strategy sets A_1 and A_2,and payoffs $u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2))$, $i = 1, 2$. Denote the NE in pure strategies by (a_1^*, a_2^*).

Subgame perfect equilibrium SPE: $(a_1^*, a_2^*, a_3^*(a_1, a_2), a_4^*(a_1, a_2))$

This framework and the concept of subgame perfection can be extended to an arbitrary number of persons and stages (see lecture 9)

Example: Banking system

Basic asymmetry of the banking system: Deposits can be withdrawn on short term basis. But if loans for investments are withdrawn on a short term basis, even "healthy" creditors go bankrupt. This holds also for the interbanking market.

$\Rightarrow$ Systemic risk of the banking system, need for regulation.

Very simple model of this basic asymmetry, and its consequences

2 investors, each of them deposits D at a bank that finances a project.

The project's profits depend on when it is liquidated:

Liqudation value in stage 1: $2R_1$, with $2D > 2R_1 > D$.

Liqudation value in stage 2: $2R_2$, with $R_2 > D$

Liqudation value in stage 3: $2R_3$, with $R_3 > R_2$.

$\Rightarrow$ The project has to run for at least two stages in order to recover the investment - The project is in principle profitable, but must run for at least two stages.

If at least one investor withdraws his money already in stage 1, the bank has to cancel the loan, and the project has to be liquidated. $\Rightarrow$

The bank cannot repay the deposit, and hence goes bankrupt with a liquidation value of $2R_1$.

If both investors withdraw their money simultaneously in stage 1, both get the same share of the liquidation value, i.e. both get R_1.

If only one investor withdraws in stage 1, he gets his deposit D whereas the other investor gets only what remains from the liquidation value, i.e.$2R_1 - D$.

If none of the investors withdraws in stage 1, the project reaches the second ("mature") stage. In this case, if both investors withdraw in stage 2, both get their share of the second stage liquidation profit, i.e. R_2. If only one withdraws, this investor gets his deposit D, wheras the other one gets the rest of the liquidation profit, i.e. $2R_2 - D$. If no one withdraws in stage 2, the project reaches stage 3 and both get R_3.

Calculation of the SPE

The second stage decision is only relevant if both have not witdrawn in the first stage. Hence, we only have to calulate the equilibrium of the static second stage game for the case that both have not withdrawn. In this case, the static second stage game is given by

	S_2^2		W_2^2	
S_1^2	R_3	R_3	$2R_2 - D$	D
W_1^2	D	$2R_2 - D$	R_2	R_2

with S_1^2 denoting player $1's$ second stage decision to stay, W_1^2 player $1's$ second stage decision to withdraw, etc.

This second stage static game has one NE, namely (S_1^2, S_2^2)

Stage 1: Note again that if at least one investor withdraws in stage one, the actions in stage 2 are irrelevant. Hence, only if both stay, the stage 2 NE enters the description of the first stage static game. So the first stage static game is given by

	S_2^1		W_2^1	
S_1^1	R_3	R_3	$2R_1 - D$	D
W_1^1	D	$2R_1 - D$	R_1	R_1

This first stage static game has two NE in pure strategies, namely (S_1^1, S_2^1) and (W_1^1, W_2^1).

Hence, 2 SPEs

SPE A:

$$\begin{array}{ll} \text{Stage 1:} & (S_1^1, S_2^1) \\ \text{Stage 2:} & (S_1^2, S_2^2) \end{array}$$

SPE B

$$\begin{array}{ll} \text{Stage 1:} & (W_1^1, W_2^1) \\ \text{Stage 2:} & (S_1^2, S_2^2) \end{array}$$

SPE B can be interpreted as a bank-run: Although the project would be in principle profitable, each investor is forced to withdraw his money too early, simply because the other player does the same - mutual distrust is enough to cause bankruptcy.

Under SPE B, it is also specified what happens in stage 2 if both investors would stay in stage 1, although SPE B specifies that both investors actually do withdraw in stage 1.

General feature of SPE: The subgame perfect equilibrium also describes actions which are taken under circumstances that are excluded by the same equilibrium.