Lecture 4. Mixed Stategies and the Existence of the Nash Equilibrium

NE in pure strategies need not exists

Example: Outguessing game (matching pennies)

	H_2		T_2	
H_1	-1	1	1	-1
T_1	1	-1	-1	1

To overcome these problems: Mixed strategies: Players do not choose a strategy for sure, but only a probability with which their different strategies are implemented.

Definition: Take a normal form game $G = \{S_1, S_2, ..S_n, u_1, u_2, ...u_n\}$ with player i having iK strategies ($S_i = \{s_{i1}, s_{i2}, ..s_{iK}\}$). Then a mixed strategy of player i is a probability distribution $p_i = (p_{i1}, ..p_{iK})$ with $p_i \geq 0$ and $p_{i1} + ... + p_{iK} = 1$.

Notation

S_i: set of *pure strategies* of i; $s_i \in S_i$

P_i: set of *mixed* strategies of i; a pure strategy is a degenerate mixed strategy.

$s = (s_1, ...s_n)$: pure strategy profile

$p = (p_1, p_2...p_n)$: profile of mixed strategies

p_{-i}: profile of mixed strategies for all players but i

P: set of mixed strategy profiles

$prob(s\,|p\,)$: probability that pure profile s is actually realized

$$prob(s\,|p\,) = p_1(s_1).p_2(s_2).p_3(s_3)...p_n(s_n)$$

$v_i(p)$: expected payoff of player i resulting from a profile of mixed strategies p

$$v_i(p) = \sum_{s \in S} prob(s\,|p\,).u_i(s)$$

"Payoff of a profile of mixed strategies is the sum of the payoffs of profiles of pure strategies, weighted with the probability that the pure strategy profile is actually realized."

This calculation of v_i is allowed, if u_i are von Neuman-Morgenstern utilities.

Definition: In the normal-form game G the mixed strategy profile $p^* = (p_1^*, p_2^*, ..p_n^*) \in P$ is a Nash equilibrium (NE), if for all players i it holds:

$$v_i(p_i^*, p_{-i}^*) \geq v_i(p_i, p_{-i}^*) \text{ for all } p_i \in P_i$$

Definition: The best response correspondence of player i, $BR_i(p_{-i})$ gives for each possible strategy profile of the other players, p_{-i}, the set of (possibly mixed) strategies of i which are optimal for i vis a vis p_{-i} :

$$\overline{p}_i \in BR_i(p_{-i}), \text{ if and only if } v_i(\overline{p}_i, p_{-i}) \geq v_i(p_i, p_{-i}) \text{ for all } p_i \in P_i.$$

Obviously, p^* is a NE, if and only if for all i :

$$p_i^* \in BR_i(p_{-i}^*)$$

Example - Outguessing game

	H_2	T_2
H_1	$-1 \quad 1$	$1 \quad -1$
T_1	$1 \quad -1$	$-1 \quad 1$

p_{1H} : probability, that player 1 chooses Head

p_{2H} : probability, that player 2 chooses Head

$$p_{1T} = 1 - p_{1H}$$
$$p_{2T} = 1 - p_{2H}$$

$$prob(H_1 H_2 | p) = p_{1H}.p_{2H}$$
$$prob(H_1 T_2 | p) = p_{1H}.(1 - p_{2H})$$
$$prob(T_1 H_2 | p) = (1 - p_{1H}).p_{2H}$$
$$prob(T_1 T_2 | p) = (1 - p_{1H}).(1 - p_{2H})$$

$$
\begin{aligned}
v_1(p) &= prob(H_1 H_2\,|p\,)u_1(H_1 H_2) + prob(H_1 T_2\,|p\,)u_1(H_1 T_2) \\
&\quad + prob(T_1 H_2\,|p\,)u_1(T_1 H_2) + prob(T_1 T_2\,|p\,)u_1(T_1 T_2) \\
&= p_{1H}.p_{2H}.(-1) + p_{1H}.(1 - p_{2H}).(1) \\
&\quad + (1 - p_{1H}).p_{2H}(1) + (1 - p_{1H}).(1 - p_{2H})(-1) \\
&= p_{1H}(2 - 4p_{2H}) + 2p_{2H} - 1
\end{aligned}
$$

$$
BR_1(p_{2H}) = \begin{cases} p_{1H} = 1 & \text{if } p_{2H} < \frac{1}{2} \\ p_{1H} \in [0,1] & \text{if } p_{2H} = \frac{1}{2} \\ p_{1H} = 0 & \text{if } p_{2H} > \frac{1}{2} \end{cases}
$$

$$
BR_2(p_{1H}) = \begin{cases} p_{2H} = 1 & \text{if } p_{1H} > \frac{1}{2} \\ p_{2H} \in [0,1] & \text{if } p_{1H} = \frac{1}{2} \\ p_{2H} = 0 & \text{if } p_{1H} < \frac{1}{2} \end{cases}
$$

Condition for NE $p^* = (p^*_{1H}, p^*_{2H})$:

$$p^*_{1H} \in BR_1(p^*_{2H}) \text{ and } p^*_{2H} \in BR_2(p^*_{1H}).$$

This condition fulfilled if and only if:

$$p^*_{1H} = \frac{1}{2} \text{ and } p^*_{2H} = \frac{1}{2}.$$

Theorem: Every normal form game $G = \{S_1, S_2, ..S_n, u_1, u_2, ...u_n\}$ with a finite number of players and a finite number of strategies for each player i exhibits at least one NE, possibly in mixed strategies.

Proof: see book

Observations:

There are games with NEs only in pure strategies.

Example: cooperation game (prisoners dilemma)

	C_2		NC_2	
C_1	2	2	0	3
NC_1	3	0	1	1

Proposition: If a pure strategy s_i' of player i does not survive iterated elimination of strictly dominated strategies, then in any (possibly mixed) NE p^* it holds that $p_i^*(s_i') = 0$.

"In equilibrium, we do not observe the play of an iteratevely dominated strategy - such strategies can be disregarded for the derivation of a NE".

Some games have pure strategy NE and NE in completely mixed strategies.

Example: Battle of the sexes

	FB		*FO*	
MB	2	1	0	0
MO	0	0	1	2