Lecture 3. Applications of Static Games of Complete Information

1. Cournot Duopoly

Strategic situation

homogenous good

linear market demand Q at price P

$$Q = \begin{cases} a - P & \text{if } P \leq a \\ 0 & \text{if } P > a \end{cases}$$

or

$$P = \begin{cases} a - Q & \text{if } Q \leq a \\ 0 & \text{if } Q > a \end{cases}$$

Good produced by 2 firms with no fix cost and constant marginal costs c.
$c < a$

Both firms decide simultaneously about quantity to supply, q_i and q_j

Transformation of the strategic situation into a normal form game:

2 players (firms)

strategy of player i: choice of his quantity q_i. Quantity can be any nonnegative amount $\Longrightarrow S_i = [0, \infty) = \mathbb{R}_+$

payoff-function

$$\pi_i(q_i, q_j) = \begin{cases} q_i(a - (q_i + q_j)) - q_i.c & \text{if } q_i + q_j \leq a \\ -q_i.c & \text{if } q_i + q_j > a \end{cases}$$

Derivation of the NE:

Assume that other firm chooses q_j. What is optimal for i?

$$\underset{q_i \in \mathbb{R}_+}{Max}\ \pi_i(q_i, q_j)$$

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - 2q_i - q_j - c \Longrightarrow$$

If $a - q_j - c \geq 0$: $BR_i(q_j) = \frac{a-q_j-c}{2}$

If $a - q_j - c < 0$: $BR_i(q_j) = 0$

Note: $q_i = \frac{a-q_j-c}{2}$ and $a - q_j - c \geq 0$ implies $q_i + q_j \leq a$

Note: If $q_i + q_j > a$, then $BR_i(q_j) = 0$

Best Response

$$BR_i(q_j) = \begin{cases} \frac{a-q_j-c}{2} & \text{if } a - q_j - c \geq 0 \\ 0 & \text{if } a - q_j - c < 0 \end{cases}$$

Similarly

$$BR_j(q_i) = \begin{cases} \frac{a-q_i-c}{2} & \text{if } a - q_i - c \geq 0 \\ 0 & \text{if } a - q_i - c < 0 \end{cases}$$

NE: Both firms choose their optimal quantity given the other's choice: (q_i^*, q_j^*) is a NE, if $q_i^* = BR_i(q_j^*)$ and $q_j^* = BR_j(q_i^*) \Longrightarrow$

$$q_i^* = q_j^* = \frac{a-c}{3}$$

In games where the strategy sets of all players are the same and where the strategies can be ordered from smaller to larger:

Definition: The strategies of players i and j are **strategic complements**, if the Best Response of i is increasing in j's strategy, and vice versa.

Definition: The strategies of players i and j are **strategic substitutes**, if the Best Response of i is decreasing in j's strategy, and vice versa.

Cournot Game: Strategies are strategic substitutes

Many games exhibit neither substitutes nor complements

Iterated elimination of strictly dominated strategies

Step 1) Easy to show every for both players quantities above the monopoly quantity $\frac{a-c}{2}$ are strictly dominated.

Step 2) It is easy to show that when every strategy above $\frac{a-c}{2}$ is eliminated for both players, for both players quantities below $\frac{a-c}{4}$ are strictly dominated.

Step 3) It is easy to show that when every strategy above $\frac{a-c}{2}$ and below $\frac{a-c}{4}$ is eliminated for both players, for both players quantities above $\frac{3a-3c}{8}$ are strictly dominated.

etc.

This process converges to a unique strategy, namely $\frac{a-c}{3}$

$\implies \frac{a-c}{3}$ is the unique NE

2. Betrand Duopoly

Strategic situation

2 differentiated goods produced by two different firms with no fix cost and constant marginal costs c.

demand for good i depends also on price of good j

$$q_i(p_i, p_j) = a - p_i + bp_j$$

with $0 < b < 2$ and $a > c$.

Both firms decide simultaneously about prices, p_i and p_j.

Transformation of the strategic situation into a normal form game

2 players (firms)

strategy of player i: choice of his price p_i. Price can be any nonnegative number $\Longrightarrow$

$$S_i = [0, \infty) = \mathbb{R}_+$$

payoff-functions

$$\begin{aligned} \pi_i &= q_i.p_i - q_i.c \\ &= (a - p_i + bp_j)p_i - (a - p_i + bp_j).c = \pi_i(p_i, p_j) \end{aligned}$$

Assume that other firm chooses p_j. What is optimal for i?

$$\underset{p_i \in \mathbb{R}_+}{Max}\, \pi_i(p_i, p_j)$$

$$\frac{\partial \pi_i(p_i, p_j)}{\partial p_i} = a - 2p_i + bp_j + c \implies$$

Best Response

$$BR_i(p_j) = \frac{a + bp_j + c}{2}$$

Similarly

$$BR_j(p_i) = \frac{a + bp_i + c}{2}$$

$\implies$ Strategies are complements

NE: Both firms choose their optimal price given the other's choice: (p_i^*, p_j^*) is a NE, if $p_i^* = BR_i(p_j^*)$ and $p_j^* = BR_j(p_i^*) \implies$

$$p_i^* = p_j^* = \frac{a + c}{2 - b}$$

3. Final-Offer Arbitration

Strategic situation

firm and trade union. firm wants a low, union wants a high wage.

conflict of the two groups is settled by binding arbitration (example: public sector employment).

final offer arbitration: both sides make simultaneously wage offers, w_f and w_u, and arbitrator picks one of the offers and implements it.

arbitrator has an ideal wage x, and he picks the offer which is closest to x. If the difference is the same, than he picks randomly one of the offers. x is known to firm and union (not unknown like in book).

Transformation of the strategic situation into a normal form game

2 players, firm and union

strategy of union: choice of its offer w_u. Offer can be any nonnegative number $\Longrightarrow S_u = [0, \infty) = \mathbb{R}_+$. Similarly, $S_f = \mathbb{R}_+$

payoff-functions:

$$\pi_f(w_f, w_u) = \begin{cases} -w_f & \text{if } |w_f - x| < |w_u - x| \\ \frac{1}{2}(-w_f) + \frac{1}{2}(-w_u) & \text{if } |w_f - x| = |w_u - x| \\ -w_u & \text{if } |w_f - x| > |w_u - x| \end{cases}$$

$$\pi_u(w_f, w_u) = \begin{cases} w_f & \text{if } |w_f - x| < |w_u - x| \\ \frac{1}{2}(w_f) + \frac{1}{2}(w_u) & \text{if } |w_f - x| = |w_u - x| \\ w_u & \text{if } |w_f - x| > |w_u - x| \end{cases}$$

Claim: $w_f^* = w_u^* = x$ is the only NE.

Proof:

Step 1) Proof that $w_f^* = w_u^* = x$ is a NE.

Assume that $w_f = x$. Then whatever w_u is chosen by the union, x is implemented. Hence, $w_u^* = x$ is optimal (like any other choice of w_u), and the union has no incentive to deviate.
Assume that $w_u = x$. Then whatever w_f is chosen by the firm, x is implemented. Hence, $w_f^* = x$ is optimal (like any other choice of w_f), and the firm has no incentive to deviate.

Step 2) Proof that there exists no other NE.

Assume not. Then one player, say the union, chooses in equilibrium $w_u^* \neq x$. There are four possible cases:

Case a) $w_u^* < x$, and w_u^* has a strictly positive probability to be implemented. The union has an incentive to deviate and make an alternative offer $w_u' = x$, leading to the for sure implementation of the higher wage w_u'. Hence, positive probability of the implementation of $w_u^* < x$ cannot be part of an equilibrium.

Case b) $w_u^* < x$, and w_u^* is not implemented. In this case the implemented w_f^* must be strictly larger than w_u^*. The firm has an incentive to deviate to $w_{f.}' = w_u^*$, leading to the implementaion of this strictly smaller wage $w_{f.}'$. Hence, this case cannot be an equilibrium.

Case c) $w_u^* > x$, and w_u^* has a strictly positive probability to be implemented. The firm has an incentive to deviate and make an alternative offer $w_f' = x$, leading to the for sure implementation of the lower wage w_f'. Hence, positive probability of the implementation of $w_u^* > x$ cannot be part of an equilibrium.

Case d) $w_u^* = x + \epsilon$ with $\epsilon > 0$, and w_u^* is not implemented. If the implemented wage w_f^* is such that $w_f^* > x - \frac{\epsilon}{2}$ then the firm has an incentive to deviate to $w_f' = x - \frac{\epsilon}{2}$, leading to the implementaion of the lower wage w_f'. If the implemented wage w_f^* is such that $w_f^* \leq x - \frac{\epsilon}{2}$ then the union has an incentive to deviate to $w_u' = x$, leading to the implementaion of the higher wage w_u'. Since in both cases at least one player has an incentive to deviate, this case cannot be an equilibrium.

Since the four cases are exhaustive, it is impossible that in equilibrium the union chooses $w_u^* \neq x$. The symmetric argument holds for the firm choosing $w_f^* \neq x$. Hence, the only equilibrium is given by $w_f^* = w_u^* = x$■

4. Network externalities - Natural Monopoly

Strategic situation

Two firms produce two products, a and b.

Price of both products is p (for the moment not set by firms, but just given)

n identical consumers, each of them buys at most one product

Positive network externalities: The more consumers buy good a, the more beneficial is the consumtion of good a. Same for b

Example: Operation systems of computers, game platforms, social networks etc: The more users use a certain system, the more programs/apps/games are developed

All consumers decide at the same time whether and which product to buy

Transformation of the situation into a static game

n players (consumers)

strategy of each player, $s_i \in \{a, b, r\}$, with r meaning refraining from a purchase

n_a is the number of consumers buying a, etc.

payoff-functions

$$u_i(s_1, s_2 \ldots s_n) = \left\{ \begin{array}{cc} \delta + \gamma_a(n_a - 1) - p & \text{if } s_i = a \\ \delta + \gamma_b(n_b - 1) - p & \text{if } s_i = b \\ 0 & \text{if } s_i = r \end{array} \right|$$

$\delta > 0$ measures the quality of the two products (for simplicity taken to be the same), and γ_a and γ_b the size of the network externalities.

$\gamma_a > \gamma_b > 0$: Both products have positive externalities, but more so product a.

Claim: If $p < \delta$, then there are two Nash equilibria (in pure strategies):
1. All consumers i choose $s_i^* = a$, i.e. $n_a = n$, $n_b = 0$. This leads to equilibrium payoffs of $u_i(s^*) = \delta + \gamma_a(n-1) - p > 0$
2. All consumers i choose $s_i^* = b$, i.e. $n_a = 0$, $n_b = n$. This leads to equilibrium payoffs of $u_i(s^*) = \delta + \gamma_b(n-1) - p > 0$

Proof: i) Are 1 and 2 equilibria?

Take equilibrium 1. If consumer i unilaterally deviates to his alternative strategy $s_i' = b$, his utility
$u_i(s_1, s_2., s_i'..s_n) = \delta + \gamma_b(1-1) - p = \delta - p \Longrightarrow$ Deviation not profitable;

If consumer i unilaterally deviates to his alternative strategy $s_i' = r$, his utility $u_i(s_1, s_2., s_i'..s_n) = 0 \Longrightarrow$ Deviation not profitable;

Proof for equilibrium 2 similar

ii) Are there other equilibria?

Case a: $n_a + n_b < n$ - Some consumers buy nothing. This cannot be an equilibrium, because buying something always gives a strictly positive payoff.

Case b: $n_a + n_b = n$, $n_a > 0$, $n_b > 0$ - Whole market shared between both products. In order that the a-consumers do not switch to b, it must holds that

$$\begin{aligned}
\delta + \gamma_a(n_a - 1) - p &\geq \delta + \gamma_b(n_b - 1 + 1) - p \Longrightarrow \\
\gamma_a(n_a - 1) &\geq \gamma_b n_b \Longrightarrow \\
\frac{\gamma_a}{\gamma_b} &\geq \frac{n_b}{n_a - 1}
\end{aligned}$$

To prevent b-consumers from switching:

$$\begin{aligned}
\delta + \gamma_b(n_b - 1) - p &\geq \delta + \gamma_a(n_a - 1 + 1) - p \Longrightarrow \\
\gamma_b(n_b - 1) &\geq \gamma_a n_a \Longrightarrow \\
\frac{n_b - 1}{n_a} &\geq \frac{\gamma_a}{\gamma_b}
\end{aligned}$$

Since $n_b > n_b - 1$ and $n_a > n_a - 1$, $\frac{n_b}{n_a - 1} > \frac{n_b - 1}{n_a} \Longrightarrow$ It is impossible that

both conditions are fulfilled at the same time - either a- or b-consumer have an incentive to switch.