# Lecture 11. Sequential Rationality in Games with Imperfect and Incomplete Information

## 1. Perfect Bayesian Equilibrium

Games with imperfect and incomplete information have information sets that consist of more than one node $\Rightarrow$

*SPE* is not enough to exclude noncredible threats.

Example: Noncredible threat game

2 NE: $(L, L')$ and $(R, R')$.

Both *NEs* also *SPEs*, since the only subgame is the whole game.

However, NE $(R, R')$ is implausible: To induce player 1 to choose $R$, player 2 must threat to play $R'$, if 1 chooses $L$. However, if 2 actually has to make a choice, $L'$ would always be better for 2 than $R'$ - threat of $R'$ is not credible $\Longrightarrow$

Equilibrium should not only consist of strategy combination, but also of beliefs.

## Requirements on the equilibrium

Requirement 1: At each information set, the player which moves must have a belief - a probability distribution - about which node in the information set he is. (For singleton information sets, the belief is of course trivial).

Requirement 2: Given their beliefs, the strategies must be sequentially rational - At each information set, the action taken by the player must maximize his payoff, given his belief and the subsequent strategy combination.

Requirement 3: The beliefs must be derived by Bayesian updating.

Noncredible threat game

For any belief of player 2 about where she is in her information set, she is best off by playing $L'$. Given this subsequent strategy "combination" and player $1's$ trivial belief about where he is in his own information set, player 1 is best off by playing $L \Longrightarrow$

Only $(L, L')$ fullfills Requirements 1 and 2.

For the noncredible threat game, it does not matter whether 2's beliefs are sensible or not - for any belief 2 should choose $L'$, requirement 3 not necessary. For other games, optimal actions do depend on the belief, so we have to ask what are sensible beliefs. $\Rightarrow$

Bayesian beliefs

Idea: beliefs that support an equilibrium strategy combination, must be consistent with this strategy combination

Noncredible threat game

equilibrium strategy combination $(L, L') \Longrightarrow$ belief must be such, that player 2 thinks that she is in the left node of her information set, when her information set is reached.

To get "sensible beliefs" in general: Bayesian updating

$p$ : profile of possibly mixed strategies

Examples from noncredible threat game

$$\begin{aligned} &\text{E1:} && p = (\tfrac{1}{2}L + \tfrac{1}{4}M + \tfrac{1}{4}R, R') \\ &\text{E2:} && p = (\tfrac{1}{8}L + \tfrac{7}{8}R, L') \\ &\text{E3:} && p = (R, R') \end{aligned}$$

$prob(k\,|p)$: probability, that information set $k$ is reached, when the players play $p$.

$$\begin{aligned} &\text{E1:} && prob(2's \text{ information set } |p) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ &\text{E2:} && prob(2' \text{ information set } |p) = \frac{1}{8} \\ &\text{E3:} && prob(2's \text{ information set } |p) = 0 \end{aligned}$$

$prob(x\,|p\,)$: probability, that node $x$ is reached, when the players play $p$

E1: $prob(2's \text{ left node } |p\,) = \frac{1}{2} \ldots.. prob(2's \text{ right node } |p\,) = \frac{1}{4}$

E2: $prob(2's \text{ left node } |p\,) = \frac{1}{8} \ldots.. prob(2's \text{ right node } |p\,) = 0$

E3: $prob(2's \text{ left node } |p\,) = 0 \ldots.. prob(2's \text{ right node } |p\,) = 0$

$b_i(x\,|p, k\,)$ : player $i's$ belief that he is in node $x$ of his information set $k$, if the information set $k$ is actually reached and the players play $p$.

Bayesian updating:

$$b_i(x\,|p, k\,) = \begin{array}{cc} \frac{prob(x|p\,)}{prob(k|p\,)} & \text{if } prob(k\,|p\,) > 0 \\ \text{anything } \in [0, 1] & \text{else} \end{array}$$

E1:

$$b(2's \text{ left node } |p, \text{2's information set}) = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

$$b(2's \text{ right node } |p, \text{2's information set}) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

E2:

$$b(2's \text{ left node } |p, \text{2's information set}) = \frac{\frac{1}{8}}{\frac{1}{8}} = 1$$

$$b(2's \text{ right node } |p, \text{2's information set}) = 0$$

E3:

$$b(2's \text{ left node } |p, \text{2's information set}) = ?$$

$$b(2's \text{ right node } |p, \text{2's information set}) = ?$$

Definition: A Perfect Bayesian equilibrium consists of a strategy combination and beliefs that fulfill Requirements 1-3.

# 2. Signaling games

Two players, Sender and Receiver

Nature draws a type $t_i$ for the Sender from a set of feasible types $T$ according to probability distribution $q(t_i)$

Sender observes $t_i$, and then decides to send a signal $m_j$ sends a signal from a set of feasible signals $M$.

Receiver observes $m_j$ (but not $t_i$), and then chooses an action $a_k$ from his action space $A$.

payoffs

$$u_S(t_i, m_j, a_k)$$
$$u_R(t_i, m_j, a_k).$$

This game used for the analysis of many different informational problems. E.g. Sender is a firm that needs money for a project. Investor uninformed about the profitability of the project, but firm can signal the profitability by its capital structure.

Other example: University Education as Job Market Signaling Device

Receiver: firm

Sender: employee

nature chooses whether a future employee is highly talented (type $h$) or not (type $l$); $q$ is the probability of $h$, $0 < q < 1$.

The future employee chooses whether take a university education $(m_e)$ or not $(m_n)$.

The firm does not observe the talent of the employee, but his university education (or the lack of it).

There are in principle two positions available for the employee: One that gives a high salary and requires high talent, and a normal one for which the talent level is irrelevant. The firm decides about whether to put the employee on the high talent position (choice $a$) or on the normal position (choice $b$).

payoffs

Employee: The high talent job pays a salary of 4, the normal job a salary of 2. Successfully attending the university requires effort, which is higher for the $l$ then for the $h$ type. The monetary equivalent of these effort costs are 1 for the $h$ and 3 for the $l$ type.

Firm: For the high talent job, the firm's profit from employing an $h$ type is 4. Employing an $l$ type for this job is really a mistake, it gives no profits at all. University education increases the productivity of both types, such that profits increase by $\epsilon \geq 0$. For the normal job, neither a high talent nor the university make a difference. The profits are always 2.

Proposition: For all $\epsilon \geq 0$, $0 < q < 1$, a Bayesian perfect equilibrium $p^* = (p_S^*, p_R^*)$ is given by:

$$\begin{aligned} p_S^* &= (m_e^h, m_n^l); \\ p_R^* &= (a_e, b_n) \\ 1 &= b_R(h \,|\text{info set } e, p^*) \\ 0 &= b_R(h \,|\text{info set } n, p^*) \end{aligned}$$

Proof:

1) Beliefs in accordance with Bayesian updating?

$$\begin{aligned} prob(\text{infoset } e \,|\, p^*) &= q \\ prob(\text{infoset } n \,|\, p^*) &= 1 - q \\ prob(h \text{ in infoset } e \,|\, p^*) &= q; \\ prob(l \text{ in infoset } e \,|\, p^*) &= 0; \\ prob(h \text{ in infoset } n \,|\, p^*) &= 0; \\ prob(l \text{ in infoset } n \,|\, p^*) &= 1 - q; \end{aligned}$$

$\Rightarrow$

$$b_R(h\,|\text{info set } e, p^*) = \frac{prob(h \text{ in info set } e\,|p^*)}{prob(\text{info set } e\,|p^*)} = 1$$
$$b_R(h\,|\text{info set } n, p^*) = \frac{prob(h \text{ in info set } n\,|p^*)}{prob(\text{info set } n\,|p^*)} = 0$$

2) Players strategies sequentially rational?

Receiver:

info set $e$ : $b_R(h\,|\text{info set } e, p^*) = 1 \Rightarrow$
$u_R(p^*, b_R) = 4 + \epsilon > u_R(p_S^*, (b_e, .), b_R) = 2$

info set $n$ : $b_R(l\,|\text{info set } n, p^*) = 1 \Rightarrow$
$u_R(p^*, b_R) = 2 > u_R(p_S^*, (., a_n), b_R) = 0$

Sender:

info set $h : u_R(p^*) = 3 > u_R((m_n^h, .), p_R^*) = 2$

info set $l : u_R(p^*) = 2 > u_R((., m_e^l), p_R^*) = 1$■

Note: This results holds for all $\epsilon \geq 0$ : Even if a university education has no direct impact on the productivity, it gives the better job.

Reason: University degree serves as a signalling device, if non-monetary costs of getting the degree are high enough for the less talented persons so that they do not go for it $\Longrightarrow$

it always pays for the students if the study program makes them work hard.