Lecture 8. Topics in General Equilibrium Theory

1. Partial Equilibrium

One market in isolation - impact of other markets ignored.

Most used model in economics, e.g. IO

What are implicit assumptions of partial equilibrium analysis

1.1. The model

Partial equilibrium model: Demand and supply depends only on own price

Within a GE framework: Two-goods economy, since GE depends only on relative price.

good l with price p

good m (money, composite of all other goods); m numeraire (price of m equals 1)

Composite good: Justified by "smallness" of l vis-a-vis $m \implies$ change of p does not changes price of all the goods from which m is composed.

I consumers, i consumes x_i units of l and m_i units of m.

Quasilinear preferences; twice differentiable utility functions

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

with $\phi_i{}' > 0$, $\phi_i{}'' \leq 0$

ω_i^m initial endowment of i in m

for simplicity: no initial endowment of l; negative m_i possible.

J firms, produce l from m. Twice differentiable cost function $c_j(q_j), c_j{}' > 0,\ c_j{}'' \geq 0$

PMP for price p^*:

$$\underset{q_j \geq 0}{Max}\ p^* q_j - c_j(q_j)$$

Necessary and sufficient first order condition for solution

$$p^* \leq \frac{\partial c_j(q_j^*)}{\partial q_j} \text{ with equality, if } q_j^* > 0.$$

Since only two goods, supply of good l does only depend on p

UMP for price p^* and production profile q_j^*:

$$\underset{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+}{Max} m_i + \phi_i(x_i)$$

$$s.t \quad : \quad m_i + p^* x_i \leq \omega_i^m + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*))$$

In optimum budget constraint fulfilled with equality $\Longrightarrow$

Necessary and sufficient first order condition for solution m_i^*, x_i^* :

$$\frac{\partial \phi_i(x_i^*)}{\partial x_i} \leq p^* \text{ with equality, if } x_i^* > 0.$$

Because of two goods and quasilinearity, demand for good l depends only on p (not on θ_{ij} or ω_i^m)

Market clearing condition:

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

demand and supply depend only on $p \implies$ Walrasian equilibrium allocation depends only on the price of the good l, and not on initial endowments or firm shares.

1.2. Welfare and Welfare Theorems in the Partial Equilibrium Context

Walrasian aggregate surplus: $\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c(q_j)$

Quasilinear preferences $\Rightarrow$ for given production and consumption of good l, $(\overline{x}_1, ...\overline{x}_I, \overline{q}_1, ...\overline{q}_J)$, utility possibility frontier is linear:

$$U = \left\{ (\overline{u}_1, ...\overline{u}_I) : \sum_{i=1}^{I} \overline{u}_i \leq \sum_{i=1}^{I} \phi_i(\overline{x}_i) + \sum_{i=1}^{I} \omega_i^m - \sum_{j=1}^{J} c(\overline{q}_j) \right\}$$

Proposition: $(x_1^*, ...x_I^*, q_1^*, ...q_J^*)$ is paretooptimal, iff Walrasian aggregate surplus is maximized.

$$\underset{(x_1,...x_I,q_1,...q_J)}{Max} \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c(q_j)$$

$$s.t \quad : \quad \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} q_j$$

Proof: FOCs

$$\begin{aligned} p^* &\leq c_j{}'(q_j^*) \text{ with equality, if } q_j^* > 0. \\ \frac{\partial \phi_i(x_i^*)}{\partial x_i} &\leq p^* \text{ with equality, if } x_i^* > 0 \\ \sum_{i=1}^{I} x_i^* &= \sum_{j=1}^{J} q_j^* \end{aligned}$$

These FOC's characterize the Walrasian equilibrium, and since every equilibrium allocation is paretoefficient (First Welfare Theorem), the solution to the maximization problem above is paretoefficient.

On the other hand, take any utility distribution $(u_1^*, ... u_I^*)$ connected with a paretoefficient allocation. Because of second welfare theorem and quasilinear preferences, there are transfers $(T_1, ... T_I)$ with $\sum_{i=1}^{I} T_i = 0$ such that the Walrasian equilibrium with initial endowments $(\omega_{1m} + T_1, ..., \omega_{mI} + T_I)$ leads to the utility distribution $(u_1^*, ... u_I^*)$. This equilibrium is again characterized by the FOC's above.■

Social welfare function $W(u_1, ... u_I) : \mathbb{R}^I \rightarrow \mathbb{R}$

"summarizes" social evaluation of allocations (i.e. the utility levels connected with allocations): describes "preferences of the whole society"

Strict monotonicity: $W(u_1, ... u_I)$ is strictly increasing in all its arguments $\Rightarrow$ only paretoefficient allocations maximize $W(u_1, ... u_I)$.

Proposition: If the individual preferences are quasilinear, and the social welfare function is strictly monotone, then it holds for any two feasible allocations $(x_1, ...x_I, q_1, ...q_J)$ and $(x_1', ...x_I', q_1', ...q_J')$:

$$\begin{aligned} W(u_1(x_1), ...u_I(x_I)) &\geq W(u_1(x_1'), ...u_I(x_I')) \text{ iff} \\ \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c(q_j) &\geq \sum_{i=1}^{I} \phi_i(x_i') - \sum_{j=1}^{J} c(q_j') \end{aligned}$$

If preferences are quasilinear and the social welfare function is strictly monotone, then the Walrasian aggregate surplus measures welfare.

2. Core

Does "pure" trade without a price system lead to a Walrasian equilibrium allocation?

I consumers, endowments ω_i, continuous, strictly convex and strongly monotone preferences $\succeq_i$

publicly available constant returns to scale technology $Y \subset \mathbb{R}^L$ - zero equilibrium profits.

an allocation x is feasible, if $\sum_{i=1}^{I} x_i \leq \sum_{i}^{I} \omega_i + y$ for some $y \in Y$.

Coalition $S : S \subset I, S \neq \varnothing$.

Definition: A coalition S blocks a feasible allocation $x^* = (x_1^* ... x_I^*)$ if for every $i \in S$ there is a consumption bundle x_i such that:

i) $x_i \succ x_i^*$ for all $i \in S$

ii) $\sum_{i \in S} x_i \in Y + \{\sum_{i \in S} \omega_i\}$

Definition: A feasible allocation $x^* = (x_1^* ... x_I^*)$ has the core property if there is no coalition blocking x^*.

Core: set of all allocations with the core property.

Proposition: Any Walrasian equilibrium allocation belongs to the core.

Proof: Let (x^*, y^*, p^*) be a Walrasian equilibrium, and suppose that coalition S can block the equilibrium allocation. Hence, there exists $y \in Y$ and $\{x_i\}_{i \in S}$ with $x_i \succ_i x_i^*$ for every $i \in S$ and $\sum_{i \in S} x_i = \sum_{i \in S} \omega_i + y$. Utility maximization at the equilibrium implies that $p^* x_i > p^* \omega_i$ for every i. Hence $p^* y = \sum_{i \in S} (p^* x_i - p^* \omega_i) > 0$. Because of constant returns, $p^* y^* = 0$. Hence $p^* y^* < p^* y$. Therefore, y^* does not maximize profits. But this contradicts equilibrium definition■

Converse not true.

2.1. N-replica economy

pure exchange economy with H types of consumers, endowments ω_h, continuous, strictly convex and strongly monotone preferences $\succeq_h$

N consumers of each type.

Proposition: Denote by *hn* the *n*-th individual of type *h*, and suppose that the allocation $x^* = (x^*_{11}, ...x^*_{1n}, ..x^*_{1N}, ...x^*_{H1}...x^*_{Hn}, ..x^*_{HN})$ belongs to the core. Then it holds that

$$x^*_{hn} = x^*_{hm} \text{ for all } 1 \leq n \leq m \leq N \text{ and for all } h.$$

Since in a core allocation the bundle of an individual depends only on his type, a core allocation of a $N - replica$ economy is given by $x^* = (x^*_1 ... x^*_H)$

Sequence of $N-$replica economies with $N \to \infty$.

Denote by C_N the core of the $N -$ *replica*

Proposition: If $x^* = (x_1^* ... x_H^*) \in C_N$ for all $N = 1, 2....$, then x^* is a Walrasian equilibrium allocation.

3. Cooperative Game Theory

The idea of a core extends to general situations $\longrightarrow$ Cooperative Game Theory.

set of agents ("players") I

coalitions: nonempty subsets of I; also the grand coalition I is a coalition.

outcome for grand coalition: list of utilities $u = (u_1, ...u_I)$

outcome for coalition $S : u^S = (u_i)_{i \in S}$

Definition: A nonempty, closed set $U^S \subset \mathbb{R}^S$ is a utility possibility set, if

$$u^S \in U^S \text{ and } u'^S \leq u^S \text{ implies that } u'^S \in U^S.$$

Definition: A game in characteristic form (I, V) is a set of players and a rule $V(.)$ that associates with every coalition S a utility possibility set $V(S) \subset \mathbb{R}^S$.

Example: Economy with constant returns to scale technology Y and I consumers with continuous, increasing, concave utility functions u_i :

$$\begin{aligned} V(S) &= \left\{ (u_i(x_i))_{i \in S} : \sum_{i \in S} x_i \right. \\ &\left. \leq \sum_{i \in S} \omega_i + y, y \in Y \} + (-\mathbb{R}^S_+) \right\} \end{aligned}$$

Definition: A game in characteristic form (I, V) is superadditive if for any coalitions S, T with $S \cap T = \varnothing$ it holds: $u^S \in V(S)$ and $u^T \in V(T)$ implies $(u^S, u^T) \in V(S \cup T)$.

Definition: A transfer utility game in characteristic form (TU-Game) is defined by (I, v), where I is a set of players and $v(.)$ a function. This characteristic function $v(.)$ assigns to every coalition S a number - the worth of S - which gives the maximum sum of utilities of the members of the coalition S.

Note: every TU-game is a game in characteristic form.

Definition: For a game in characteristic form (I, V) the utility outcome $u = (u_1 ... u_I) \in V(I)$ is blocked by coalition $S \subset I$, if there exists $u'^S \subset V(S)$ such that $u'^S > u^S$ for all $i \in S$.

Definition: A utility outcome $u = (u_1 ... u_I) \in V(I)$ belongs to the core of the game in characteristic form (I, V), if there is no blocking coalition.

In general the core can be empty or can consist of many allocations.

Extension of the concept of core with equity considerations $\Longrightarrow$

Shapley value (only TU-case considered):

Definition: A subgame (S, v) of a TU-game (I, v) is defined by $S \subset I$ and the restriction of $v(.)$ to the subsets of S.

Definition: A family of numbers $\{Sh_i(S,v)\}_{S\subset I, i\in I}$ is an egalitarian solution, if for every subgame (S,v) and players $i,h\in S$ the following holds:

$$\begin{aligned} Sh_i(S,v) - Sh_i(S\backslash\{h\},v) &= Sh_h(S,v) - Sh_h(S\backslash\{i\},v) \\ \sum_{i\in S} Sh_i(S,v) &= v(S) \end{aligned}$$

Definition: The Shapley value of (I,v), denoted by $Sh(I,v) = (Sh_1(I,v), ...Sh_I(I,v))$, is an outcome consistent with the egalitarian solution.

Proposition: For all TU-games there exists a unique Shapley value.

Definition: A TU-game (I, v) is convex if $S \subset T$ and $i \in I \setminus T$, then

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Proposition: For any convex TU-game the Shapley value belongs to the core.