Lecture 7: General Equilibrium - Existence, Uniqueness, Stability

In this lecture: Preferences are assumed to be rational, continuous, strictly convex, and strongly monotone.

1. Excess demand function

Pure exchange economy: $Y = -\mathbb{R}^L_+ \Rightarrow$ No production, no profits

$x_i(p, w_i)$ Walrasian demand function of consumer i for wealth w_i and price p

$\Rightarrow x_i(p, p\omega_i)$ Walrasian demand of i in a pure exchange economy

Note: Due to strict convexity, $x_i(p, p\omega_i)$ is single-valued.

Excess demand function of i, $z_i(.) : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}^L$

$$z_i(p) = x_i(p, p\omega_i) - \omega_i$$

Aggregate excess demand function, $z(.) : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}^L$

$$z(p) = \sum_{i=1}^{I} z_i(p)$$

Obviously, a Walrasian equilibrium is an allocation (x^*) and a price vector p^*, such that:

$$z(p^*) = 0$$

Proposition: Take a pure exchange economy with consumer's preferences being rational, continuous, strictly convex, and strongly monotone, and $\omega >> 0$. The aggregate excess demand function satisfies the following properties:

i) $z(.)$ is continuous.

ii) $z(.)$ is homogenous of degree zero in the price p.

iii) $pz(p) = 0$ for all p (Walras law).

iv) There is an $s > 0$ such that $z_l(p) > -s$ for all l and all p

v) If $p^n \rightarrow p$ with $p \neq 0$ and $p_l = 0$ for some l, then

$$Max\{z_1(p^n), ..z_L(p^n)\} \rightarrow \infty$$

Proof: i)-iii) follow directly from the properties of the demand function. iv) follows from the finiteness of the initial endowment, and v) follows from strong monotonicity.

Economy with production:

$\pi_j(p), y_j(p)$ maximum profit and profit maximizing production vector under price p.

Assume that $y_j(p)$ is single valued (decreasing returns to scale)

$x_i(p, w_i)$ Walrasian demand function of consumer i for wealth w_i and price $p \Rightarrow$

$x_i\left(p, p\omega_i + \sum_{j=1}^{J} \theta_{ij}\pi_j(p)\right)$ Walrasian demand of i in an economy with production

Aggregate excess demand function:

$$z(p) = \sum_{i=1}^{I} x_i\left[p, p\omega_i + \sum_{j=1}^{J} \theta_{ij}\pi_j(p)\right] - \sum_{i=1}^{I} \omega_i - \sum_{j=1}^{J} y_j(p)$$

Obviously, a Walrasian equilibrium is an allocation (x^*, y^*) and a price vector p^*, such that:

$$z(p^*) = 0$$

Proposition: Take a private ownership economy $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(\omega_i, \theta_{i1}...\theta_{iJ})\}_{i=1}^{I})$. Assume that the preferences are rational, continuous, strictly convex, and strongly monotone. The production sets are closed, strictly convex, and bounded above. Furthermore, there exists an $x >> 0$ with $x \in (\omega + Y)$. Then the aggregate excess demand function satisfies the properties:

i) $z(.)$ is continuous.
ii) $z(.)$ is homogenous of degree zero in the price p.
iii) $pz(p) = 0$ for all p (Walras law).
iv) There is an $s > 0$ such that $z_l(p) > -s$ for every l and all p
v) If $p^n \rightarrow p$ with $p \neq 0$ and $p_l = 0$ for some l, then

$$Max\{z_1(p^n), ..z_L(p^n)\} \rightarrow \infty$$

2. Existence of a general equilibrium

Brouwer's fixed point theorem:
Suppose that $A \subset \mathbb{R}^N$ is closed, bounded, and convex. $f : A \longrightarrow A$ is a continuous function. Then $f(.)$ has a fixed-point $a^* \in A$ such that $f(a^*) = a^*$.

Proposition: Take an aggregate excess demand function $z(p)$ that fulfills all the properties of the previous proposition. Then there exists a price vector p^* such that $z(p^*) = 0$.

Proof: Because of homogeneity, we restrict attention to price vectors that belong to the $(L-1)$ - dimensional unit simplex $\Delta^{L-1} = \{p \in \mathbb{R}_+^L : \sum_l p_l = 1\}$. Note that Δ^{L-1} is closed, bounded, and convex.

Step 1: There exists $p^* \in \Delta^{L-1}$ such that $z(p^*) \leq 0$:

Assume to the contrary that there exists no $p^* \in \Delta^{L-1}$ such that $z(p^*) \leq 0 \Rightarrow$ for each $p \in \Delta^{L-1}$ there exists an l with $z_l(p) > 0$

Let $z^+(.)$ be given by $z_l^+(p) = Max\{z_l(p), 0\} \Rightarrow$ for each $p \in \Delta^{L-1}$, $\sum_l z_l^+(p) > 0$

Let $f : \Delta^{L-1} \to \Delta^{L-1}$ with $f(p) = \frac{p+z^+(p)}{1+\sum_l z_l^+(p)}$.

Since $z^+(.)$ is continuous, $f(p)$ is continuous $\Rightarrow$

There exist a fixed-point $\overline{p} \in \Delta^{L-1}$ such that $f(\overline{p}) = \overline{p} \Rightarrow$

$\overline{p}(1 + \sum_l z_l^+(\overline{p})) = (\overline{p} + z^+(\overline{p})) \Leftrightarrow \overline{p} = \frac{z^+(\overline{p})}{\sum_l z_l^+(\overline{p})}$

By Walras law, $0 = \overline{p}z(\overline{p}) = \frac{z^+(\overline{p})}{\sum_l z_l^+(\overline{p})} z(\overline{p}) \Rightarrow z^+(\overline{p})z(\overline{p}) = 0 \Rightarrow z(\overline{p}) \leq 0$

- a contradiction

Step 2: $p^* >> 0$.
If $p_l^* = 0$ for some l, then by v) of the previous proposition there exists some good k which has a strictly positive excess demand, contradicting step 1.

Step 3: Because of Walras law and step 2, it cannot be that $z_l(p^*) \leq 0$ for all l, and $z_l(p^*) < 0$ for some l. Hence, $z(p^*) = 0$.$\blacksquare$

Existence of equilibria can also be shown when preferences are not strongly monotone, but locally nonsatiated. Same holds for production sets which do not exhibit decreasing returns to scale and preferences which are not strictly convex, allowing for excess demand correspondences. Also for production sets which are not bounded above.

3. The number of equilibria

Because of Walras' law, the aggregate excess demand can be characterized by the demand for $L-1$ goods. Because of homogeneity, the aggregate demand depends only on normalized prices, normalized e.g. by setting one price equal to 1. Hence, $\widehat{z}(p) : \mathbb{R}_{+}^{L-1} \rightarrow \mathbb{R}^{L-1}$ fully characterizes the economy.

Let $D\widehat{z}(p)$ be the$(L-1)x(L-1)$ matrix of price effects at normalized price p.

$$D\widehat{z}(p) = \begin{array}{cccc} \frac{\partial \widehat{z}_1(p)}{\partial p_1} & \frac{\partial \widehat{z}_2(p)}{\partial p_1} & . & . \\ . & . & \frac{\partial \widehat{z}_m(p)}{\partial p_n} & . \\ . & . & . & . \\ . & . & . & \frac{\partial \widehat{z}_{L-1}(p)}{\partial p_{L-1}} \end{array}$$

Definition: An equilibrium price vector $p^* = (p_1^* ... p_{L-1}^*)$ is regular, if $D\widehat{z}(p^*)$ has full rank $(L-1)$. If every normalized equilibrium price vector is regular, the economy is regular.

Non-regular economies are non-generic ("untypical ")

Proposition: For any regular economy, the number of equilibria is finite and odd.

Intuition: for very low relative prices, excess demand is positive. For high proces, excess demand is negative. Inbetween are equilibria with excess demand equals zero. Generically, a curve that starts above zero and ends below zero crosses the zero-line an odd number of time.

4. Uniqueness

4.1.Constant returns to scale technologies

Definition: The excess demand function $z(.)$ satisfies the weak axiom of revealed preference, if for any price vectors p, p' we get:

$$z(p) \neq z(p') \text{ and } pz(p') \leq 0 \text{ implies } p'z(p) > 0.$$

Proposition: If $z(.)$ satisfies the weak axiom, then for any constant returns to scale technology Y the set of equilibrium price vectors is convex. If in addition the economy is regular, the equilibrium is unique.

4.2. Gross substitutes

Definition: The excess demand function $z(.)$ has the gross substitution property if for any p, p' with $p'_l > p_l$ for some l and $p'_k = p_k$ for all $k \neq l$, it holds that $z_k(p') > z_k(p)$ for all $k \neq l$.

Proposition: If $z(.)$ is the aggregate demand function of a pure exchange economy, and it has the gross substitution property, then this economy has at most one equilibrium.

4.3. Initial endowment

Proposition: Suppose that in a pure exchange economy the initial endowment allocation $\overline{\omega} = (\overline{\omega}_1 ... \overline{\omega}_I)$ is a Walrasian equilibrium allocation with strictly convex and strongly monotone preferences. Then this is the unique equilibrium allocation.

5. Stability

What happens if prices are not at equilibrium level?

Walras: Sign of change of price of a good is the same as the sign of the excess demand of this good.

$\frac{dp_l}{dt} = c_l z_l(p)$, with c_l being a strictly positive constant, and p being a relative (normalized) price.

Of course, in equilibrium $\frac{dp_l}{dt} = 0$. But a relative equilibrium price needs not to be globally stable (for any initial relative price, the system converges to the equilibrium price) nor even locally stable (whenever the initial relative price is sufficiently close to relative equilibrium price, the relative price converges to the equilibrium level)

Proposition: Suppose that $z(p^*) = 0$ and $p^* z(p) > 0$ for every p not proportional to p^*. Then the relative prices of any solution trajectory of the price tâtonnement converge to the relative prices of p^* - p^* is globally stable.