# Lecture 3. Production Theory

## 1. Production sets

economy with $L$ goods

production vector $y \in \mathbb{R}^L$ (production plan)

negative components: inputs
positive components: outputs

production set $Y \subset \mathbb{R}^L$ : set of feasible production vectors, describes technology

Transformation function $F : \mathbb{R}^L \to \mathbb{R}$ such that
$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$ and $F(y) = 0$ if $y$ is element of the boundary of $Y$.

For differentiable $F$ and $\overline{y} : F(\overline{y}) = 0$ : Marginal rate of transformation:

$$MRT_{lk}(\overline{y}) = \frac{\partial F(\overline{y})/\partial y_l}{\partial F(\overline{y})/\partial y_k}$$

Technologies with distinct inputs and outputs: set of outputs distinct from set of inputs.

$q = (q_1, q_2 ... q_m)$ : output vector, production level

$z = (z_1, z_2 ... z_{L-m})$: input vector (now measured in *nonnegative* numbers)

Single output: Production function $f(z)$

$$MRT_{lk}(\overline{z}) = \frac{\partial f(\overline{z})/\partial z_l}{\partial f(\overline{z})/\partial z_k}$$

Properties of production sets:

$Y$ is nonempty

$Y$ is closed: $y^n \to y$ and $y^n \in Y$ for all $n$ implies $y \in Y$.

No free lunch: If $y \in Y$ and $y \geq 0$, then $y = 0$.

Inaction possible: $0 \in Y$

Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$

Irreversibility: If $y \in Y$ and $y \neq 0$, then $-y \notin Y$

Additivity (free entry): If $y \in Y$ and $y' \in Y$, then $y + y' \in Y$

Convexity: If $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha)y' \in Y$

Returns to scale:

Nonincreasing RTS: If $y \in Y$ and $\alpha \in [0, 1]$, then $\alpha y \in Y$.

Nondecreasing RTS: If $y \in Y$ and $\alpha \geq 1$, then $\alpha y \in Y$.

Constant returns to scale: If $y \in Y$ and $\alpha \geq 0$, then $\alpha y \in Y$.

# 2. Profit Maximization and Cost Minimization

Profit Maximization Problem (PMP):

$$\max_y \; py$$
$$s.t \quad : \quad y \in Y$$

equivalently:

$$\max_y \; py$$
$$s.t \quad : \quad F(y) \leq 0$$

Price taking behavior assumed.

Profit function $\pi(p)$ : the value of the solution of the PMP for given prices

Supply correspondence $y(p)$: the set of production vectors that maximize profits for given prices

Proposition: If $Y$ exhibits nondecreasing returns to scale, then $\pi(p)$ is either nonpositive or infinity.

Proposition: Suppose that $Y$ is closed and satisfies free disposal. Then the profit function $\pi(p)$ and the supply correspondence $y(p)$ have the following properties:

i) $\alpha\pi(p) = \pi(\alpha p)$ for all $\alpha > 0$.

ii) $\pi(p)$ is convex.

iii) If $Y$ is convex, then $Y = \{y \in \mathbb{R}^L : py \leq \pi(p) \text{ for all } p >> 0\}$

iv) $y(p) = y(\alpha p)$ for all $\alpha > 0$.

v) If $Y$ is convex, then $y(p)$ is a convex set for all $p$. If $Y$ is strictly convex, then $y(p)$ is single-valued.

vi) Law of supply: $(p - p')(y(p) - y(p')) \geq 0$

Single-output technology - Output good 1

$p$ : price of output

$w$ : vector of input prices

PMP:

$$Max_{z\geq 0}\ pf(z) - wz$$

If $f(z)$ is differentiable and the technology exhibits nonincreasing returns to scale, then the first order conditions for profit maximization are for all $l = 2...L$ given by:

$$p\frac{\partial f(z^*)}{\partial z_l} \leq w_l, \text{ with equality if } z_l^* > 0.$$

Cost minimization problem (CMP)

$$\min_{z \geq 0} \; wz$$
$$s.t \quad : \quad f(z) \geq q$$

Cost function $c(w, q)$: minimal cost to produce $q$ for input price vector $w$.

Conditional factor demand correspondence $z(w, q)$

Proposition: Suppose that $Y$ is closed, satisfies free disposal, and describes a single-output technology. Then the cost function $c(w,q)$ and the conditional factor demand correspondence $z(w,q)$ have the following properties:

i) $c(.)$ is homogenous of degree one in $w$ and nondecreasing in $q$.

ii) $c(.)$ is concave in $w$.

iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every $q$, then $Y = \{(-z, q) : wz \geq c(w, q) \text{ for all } w >> 0\}$

iv) $z(.)$ is homogenous of degree zero in $w$.

v) If $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w,q)$ is convex. If $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w,q)$ is single-valued.

vi) If the technology exhibits constant returns to scale, then $c(.)$ and $z(.)$ are homogenous of degree 1 in $q$.

vii) If $f(.)$ is concave, then $c(.)$ is convex in $q$.

# 3. Aggregation

$J$ production units (e.g. firms) with $Y_j$, $\pi_j(p)$, and $y_j(p)$ for all $j = 1, 2...J$.

Aggregate supply correspondence $y(p)$

$$y(p) = \{y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j \text{ for some } y_j \in y_j(p)\}$$

Aggregate production set $Y$

$$Y = Y_1 + Y_2 + ....Y_J = \{y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j \text{ for some } y_j \in Y_j\}$$

$\pi^*(p)$ denotes the profit function and $y^*(p)$ the supply correspondence of $Y$.

Proposition: For all $p >> 0$, it holds:

i) $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$

ii) $y^*(p) = \sum_{j=1}^{J} y_j(p) = \{\sum_{j=1}^{J} y_j : y_j \in y_j(p) \text{ for every } j\}$

Proof:

i) Denote by $y_j$ any individual profit maximizing production plan of firm $j$. The sum of the individual profit maximizing production plans is an element of the aggregate production set, i.e. $\sum_{j=1}^{J} y_j \in Y$. Hence $\pi^*(p) \geq p \sum_{j=1}^{J} y_j = \sum_{j=1}^{J} py_j = \sum_{j=1}^{J} \pi_j(p)$. On the other hand, consider any $y \in Y$. By definition of $y$ there exist $y_1, y_2, ...y_J$ such that $\sum_{j=1}^{J} y_j = y$.

Hence $py = p \sum_{j=1}^{J} y_j = \sum_{j=1}^{J} py_j \leq \sum_{j=1}^{J} \pi_j(p)$. So $\pi^*(p) \leq \sum_{j=1}^{J} \pi_j(p)$.

ii) Denote by $y_j$ any individual profit maximizing production plan of firm $j$. Then $p \sum_{j=1}^{J} y_j = \sum_{j=1}^{J} py_j = \sum_{j=1}^{J} \pi_j(p) = \pi^*(p)$. Hence $\sum_{j=1}^{J} y_j \in y^*(p)$, and $\sum_{j=1}^{J} y_j(p) \subset y^*(p)$. On the other hand, consider any $y \in y^*(p)$. By definition of $y$ there exist $y_1, y_2, ...y_J$ such that $\sum_{j=1}^{J} y_j = y$. Since $p \sum_{j=1}^{J} y_j = \pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$, and since for every $j$ $py_j \leq \pi_j(p)$, it must be that $py_j = \pi_j(p)$ for every $j$. Hence, $y_j \in y_j(p)$, so $y^*(p) \subset \sum_{j=1}^{J} y_j(p)$.

## 4. Efficient Production

Definition: A production vector $y$ is efficient if there does not exist $y' \neq y$ such that $y' \geq y$.

Proposition:

i) If $y \in Y$ is profit maximizing, it is also efficient.

ii) If $Y$ is convex, then every efficient production plan $y$ is profit maximizing for some $p >> 0$.