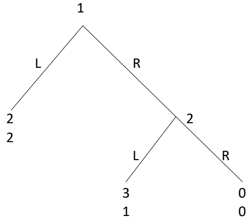
Lecture 12: Equilibrium Refinements for Extensive Form Games

Extensive form games have many many Nash-equilibria, some not plausible ("incredible threats")

Example:



	L		R	
L	2	2	2	2
R	3	1	0	0

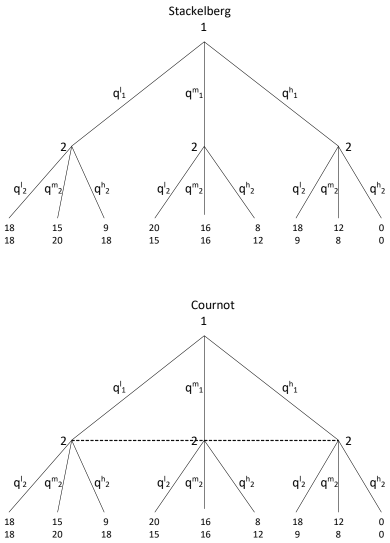
NE: (R, L) and (L, R), but (L, R) unplausible

1. Subgame Perfection

Definition: A subgame G of an extensive form game T consists of a single node x and all its successors in T, with the property that if $x \in G$ and $x' \in h(x)$, then $x' \in G$. x and x' are in the same information set in G iff they are in the same information set in T. The payoff-function of G is the restriction of the original payoff-function of T to those terminal nodes belonging to G.

Note: Every extensive form game is a subgame of itself.

Example:Cournot vs Stackelberg



Let b_i be a behavior strategy of i for game T, and let $\widehat{H}$ be the set of information sets belonging to a subgame G. Then the restriction of b_i to G is $\widehat{b}_i$ with $\widehat{b}_i(. \mid h_i) = b_i(. \mid h_i)$ for all $h_i \in \widehat{H}$.

Definition: A behavior strategy profile b is subgame perfect, if the restriction of b to a subgame G is a Nash- equilibrium of G for every subgame.

Every subgame perfect equilibrium is a Nash equilibrium, but not the other way round.

Theorem: Every finite extensive form game has at least one subgame perfect equilibrium.

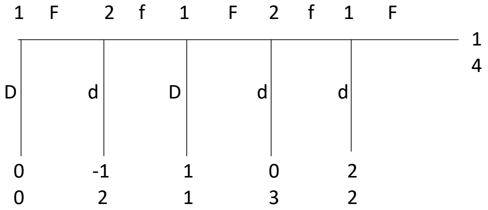
"Proof" (Application of backward induction): For any terminal node, take the subgame which contains this terminal node, and which does not contain any "smaller" subgame. Calculate the Nash-equilibria of these subgames, and create a new game by replacing each subgame by the payoffs of one of the Nash-equilibria of this subgame. For this new game, the take for any terminal node the "smallest" subgame containing the terminal node, etc. Since the game is finite, this process comes to an end after finite steps. The strategy combination consisting of the Nash-equilibrium profiles of all subgames is obviously subgame perfect.

Backward induction only feasible for finite games, but SPE also for infinite games

2. Problems with Subgame Perfection

2.1. Rationality after unexpected moves

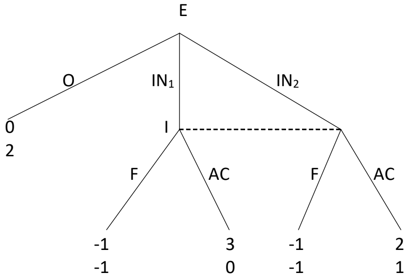
Example Centipede game



Why should player 2 believe in common knowledge of rationality when that he observes that 1 plays F at 1's first decion node?

2.2. Survival of unplausible equilibria

Example: Another market entry game



(O, F) is a NE and a SPE, but unplausible

Solution: Explicit specification of beliefs at information sets with more than one node.

3. Beliefs

Denote by i_h the player controling information set h.

Definition: A system of beliefs μ for an extensive form game Γ is a specification of a probabilities $\mu(x) \in [0,1]$ for each x in Γ such that

$$\sum_{x \in h} \mu(x) = 1$$

for all $h \in H$.

For each information set h the belief system μ specifies a probability distribution over the nodes x belonging to h.

Interpretation: For each info set h the belief system μ specifies the belief of player i_h at which node within the info set he is when h is actually reached.

Whenever h consists of only one node, there is trivially only one belief possible.

4. Perfect Bayesian Equilibrium

Definition: A belief system μ is consistent with a strategy profile b if it holds for all x, $x \in h_x$, that

$$\mu(x) = \frac{prob(x\,|b)}{prob(h_x\,|b)}$$

whenever $prob(h_x\,|b) > 0$.

"Whenever the strategy profile b does not exclude that a particular info set h_x is reached, the belief of the player i_{h_x} is consistent with Bayesian updating".

Recall: $prob(h\,|b) = \sum_{x \in h} prob(x\,|b)$

Definition: A stategy profile b is sequentially rational with respect to a belief system μ if for all information sets $h \in H$ the following holds:

$$E[u_{i_h} | h, \mu, b_{i_h}, b_{-i_h}] \geq E\left[u_{i_h} \middle| h, \mu, \widetilde{b}_{i_h}, b_{-i_h}\right]$$

for all $\widetilde{b}_{i_h} \in B_{i_h}$ that allow to reach h.

"Given that info set h is reached, given player i_h's belief about where in the info set he is, and given what the other players are doing, player i_h's choice at h maximizes his expected payoff."

Definition: A strategy profile b and a belief system μ form a Perfect Bayesian Equilibrium (PBE) if:

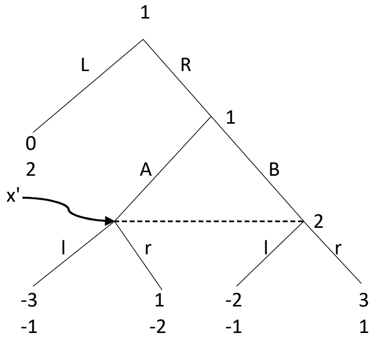
i) μ is consistent with b, and

ii) b is sequentially rational with respect to μ.

Depending on the application, PBE is sometimes too weak - does not necessarely respect subgame perfection.

Problem: No restrictions on beliefs of info sets that should not be reached - sometimes stronger requirements on the beliefs needed.

Example



(LB, l) with $\mu(x') = 1$ is PBE, but not SPE - only SPE (RB, r)

5. Sequential Equilibrium

If b_m is a completely mixed strategy profile, all info sets are reached with positive probability $\Rightarrow$

unique belief profile μ that is consistent with b_m, denoted by $\mu(b_m)$.

Let $\{b_m^k\}$ be a converging sequence of completely mixed strategy profiles with $lim_{k\to\infty} b_m^k = \overline{b}$.

Take the limit of the belief profiles consistent with the $\{b_m^k\}$, $lim_{k\to\infty}\mu(b_m^k) = \overline{\mu}$. $\overline{\mu}$ is unique and consistent with $\overline{b}$.

Definition: A strategy profile b and a belief system μ form a Sequential Equilibrium if:

i) b is sequentially rational with respect to μ.

ii) There exists a sequence of completely mixed strategy profiles $\{b_m^k\}$ such that $lim_{k\to\infty} b_m^k = b$ and $lim_{k\to\infty} \mu(b_m^k) = \mu$.