Lectures 11. Extensive Form Games

Normal form games: all strategy choices simultaneously

Extensive form games: dynamic structure of choices

Example: Stackelberg game (figure 3.3.)

1. Definition of an extensive form game

A finite extensive form game Γ consists of 6 elements:

1) I : set of players (finite, until noted otherwise); typical element i; "player" 0 ("nature") to model games with incomplete information

2) game tree:

finite set of nodes X, with a typical element x.

a precedence relation $\succ$ on $X : x \succ y$: "node x is before node y."

$\succ$ is transitive and asymmetric (not complete)

initial node 0 before all other nodes, where player 0 moves.
games with complete information: nature has only one move, hence 0 often not depicted.

each node x has exactly one immediate predecessor x': for each $x \in X \backslash \{0\}$ there exists $x' \in X$ such that $x' \succ x$, and if $x'' \succ x$, then either $x' = x''$ or $x'' \succ x'$.

set of endnodes $Z = \{z \in X : \nexists x \in X \text{ with } z \succ x\}$.

control of nodes: $i : X \backslash (Z \cup \{0\}) \rightarrow I$, with $i(x)$ being the player who controls x.

3) payoff-functions:

payoff-function of $i : u_i : Z \rightarrow \mathbb{R}$.

4) actions of $i(x)$:

A: set of actions (typically finite)

$l : X \backslash \{0\} \rightarrow A$; $l(x)$ last action taken to reach x

If x and x' are immediate successors of the same node x" with $x \neq x'$, then $l(x) \neq l(x')$

$A(x)$: set of feasible actions at x with

$$\begin{aligned} x &\in Z \Longrightarrow A(x) = \varnothing \\ x &= 0 \Longrightarrow |A(x)| \geq 1 \\ x &\notin Z \cup \{0\} \Longrightarrow |A(x)| \geq 2 \end{aligned}$$

5) information of players:

information sets $h \in H$, with H being a partition of $X \backslash (Z \cup \{0\})$ (each node x belongs to exactly one information set h).

If $x' \in h$ and $x \in h$, then player does not know whether he is at x' or x (or any other node $x'' \in h$)

This implies:

$$\begin{aligned} x, x' &\in h \Longrightarrow i(x) = i(x'). \\ x, x' &\in h \Longrightarrow A(x) = A(x'). \end{aligned}$$

$H_{\bar{i}} = \{h \in H\colon\ i(x) = \bar{i} \text{ for } x \in h\}$

6) probability distribution over $A(0)$: probability, with which nature "chooses " its different options.

Games of complete information: $|A(0)| = 1$

Games of perfect information: all information sets are singletons.

Information structure determines sequentiality

Example: Prisoners' Dilemma

perfect recall: No player forgets what he previously knew, and no player forgets his own actions.

perfect recall nearly always assumed

Example: Absent-minded driver

2. Strategies and Behavior Strategies

$A(h_i) = \{a \in A(x) \text{ for } x \in h_i\}$

pure strategy of $i, s_i : H_i \to A_i$ with $s_i(h_i) \in A(h_i)$ for all h_i

set of all pure strategies of i, $S_i = \prod_{h_i \in H_i} A(h_i)$

probability distribution over $A(0)$ plus a combination of pure strategies for all players induces prob. distribution over endnodes, and hence over the payoffs attributed to the endnodes.

payoff of a pure strategy combination = expected payoff

$\Delta(A(h_i))$ set of probability distributions over $A(h_i)$

Behavior strategy ("mixed strategy of extensive form game") b_i: for each infoset h_i controlled by i a probability distribution over the actions feasible at h_i

$$b_i \in \prod_{h_i \in H_i} \Delta(A(h_i)) = B_i$$

Again, combination of behavior strategies plus prob. distribution over $A(0)$ induces prob.distribution over payoffs attributed to the endnodes

payoff of behavioral strategy profile = expected payoff of this distribution.

3. Normal-Form Representation of Extensive Form Games

Normal form representation of an extensive form game: To define the normal form representation, let the set of players, the sets of pure strategies, and the payoffs of pure strategy profiles be the same as in the extensive form game.

Example: Sequential prisoners dilemma

Different extensive form games are represented by same normal form

Normal-form representation has "unnecessary" strategies

Definition: In a normal-form game two pure strategies are equivalent, if they induce the same probability distributions over the outcomes for all pure strategies of the opponents.

Definition: The reduced normal form game of an extensive form game is obtained by eliminating all but one strategy for each class of equivalent strategies.

$R_i(h_i)$: set of pure strategies of player i that do not preclude h_i

The mixed strategy σ_i of the normal form representation generates a unique behavior strategy b_i as follows:

If the support of σ_i contains some $s_i \in R_i(h_i)$:

$$b_i(a_i \,|h_i) = \frac{\sum\limits_{\{s_i \in R_i(h_i) \text{ and } s_i(h_i) = a_i\}} \sigma_i(s_i)}{\sum\limits_{\{s_i \in R_i(h_i)\}} \sigma_i(s_i)}$$

If the support of σ_i does not contain any $s_i \in R_i(h_i)$:

$$b_i(a_i \,|h_i) = \sum_{\{s_i(h_i) = a_i\}} \sigma_i(s_i)$$

Obviously, $b_i(a_i \,|h_i) \geq 0$ and $\sum\limits_{a_i \in A(h_i)} b_i(a_i \,|h_i) = 1$ for all h_i.

Theorem (Kuhn 1953): In a game with perfect recall, every mixed strategy σ_i is equivalent to the unique behavior strategy b_i generated by σ_i, and each behavior strategy b_i is equivalent to any mixed strategy σ_i that generates b_i.

"For any extensive form game with perfect recall it does not matter, whether the extensive form game or its normal form representation is analyzed".

4. Nash-equilibrium

Definition: A Nash-equilibrium in behavior strategies is a profile of behavior strategies, b, such that no players i has an alternative behavior strategy b_i' that gives i a strictly larger payoff than b_i when the other players play b_{-i}.

Existence of the Nash-equilibrium in behavior strategies assured by the existence theorem of lecture 9 and the equivalence of mixed and behavior strategies.

Zermelo-Theorem: A finite game of perfect information has (at least) one pure-strategy Nash equilibrium.