Literature

M: Mas-Colell, Whinston and Greene: Microeconomic Theory, Oxford University Press
F: Fudenberg and Tirole: Game Theory, MIT Press
CD: Carlsson, H. and E. van Damme, Global Games and Equilibrium Selection, Econometrica 1993, 989-1018.

Structure of the Course

A: Individual Decision Theory

1. Consumer Theory (M1,2,3)
2. Consumer Theory (M3,4)
3. Production Theory (M5)

4+5. Choice under Risk and Uncertainty (M6)

B. Market Equilibrium

6. General Equilibrium: Definition and Welfare Properties (M16)
7. General Equilibrium: Existence, Uniqueness, Stability (M17)
8. Topics in General Equilibrium Theory: Partial Equilibrium (M10), Core and Cooperative Game Theory(M18)

C. Non-cooperative Game Theory

9. Normal Form Games: Complete Information (F 1,2)
10. Normal Form Games: Incomplete Information (F6)
11. Extensive Form Games (F3,4)
12. Equilibrium Refinements for Extensive Form Games (F3,F4,M9)

12a. Global Games (CD)

Lecture 1. Consumer Theory

1. Preferences

L goods $\Longrightarrow$

consumption set $X \subseteq \mathbb{R}_+^L$

$x \in X$: consumption bundle (vector)

Preferences: Binary relation $\succeq$ on X

Strong preference and indifference: For all $x, y \in X$:

$(x \succeq y)$ and $(y \succeq x) \Longleftrightarrow x \sim y$

$(x \succeq y)$ and not $(y \succeq x) \Longleftrightarrow x \succ y$

Properties of $\succeq$:

Completeness: For all $x, y \in X$: Either $(x \succeq y)$ or $(y \succeq x)$, or both.

Transitivity: For all $x, y, z \in X$: $(x \succeq y)$ and $(y \succeq z)$ implies $(x \succeq z)$.

Completeness and transitivity: rational preferences

Weak monotonicity: $x, y \in X$ with $y >> x$ implies $(y \succ x)$

Strong monotonicity: $x, y \in X, y \neq x, y \geq x$ implies $(y \succ x)$

Local nonsatiation: For all $x \in X, \epsilon > 0$, there exists $y \in X$ such that $(y \succ x)$ and $\|y - x\| < \epsilon$

strong monotonicity implies weak monotonicity implies local nonsatiation

"Goods are indeed goods" - local nonsatiation normally assumed

Convexity: $(y \succeq x)$ and $(z \succeq x) \Longrightarrow (\alpha y + (1-\alpha)z) \succeq x$ for any $\alpha \in [0,1]$

Strict Convexity: $z \neq y, (y \succeq x)$ and $(z \succeq x) \Longrightarrow (\alpha y + (1-\alpha)z) \succ x$ for any $\alpha \in (0,1)$

Continuity: $\succeq$ is continuous, iff for any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succeq y^n$ for all n it holds:

$$\lim_{n\to\infty} x^n \succeq \lim_{n\to\infty} y^n$$

Important special cases:

Homothethic preferences: $(y \sim x) \Longrightarrow \alpha y \sim \alpha x$ for any $\alpha > 0$.

Quasilinear preferences: $\succeq$ is quasi-linear with respect to good 1, if:

i)$(y \sim x) \Longrightarrow (x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, 0..0)$ and any $\alpha \in \mathbb{R}$.
ii)$(x + \alpha e_1) \succ x$ for $e_1 = (1, 0, 0..0)$ and any $\alpha > 0$

2. Utility

A function $u : X \to \mathbb{R}$ represents a preference relation $\succeq$, iff for all $x, y \in X$ with $y \succeq x$, it holds that $u(y) \geq u(x)$.

Lemma: If u represents a particular preference relation $\succeq$, and v is a monotone transformation of u, then $\succeq$ is also represented by v.

Example: $v(u(x)) = u(x)^3$

Proposition: If $\succeq$ is rational and continuous, there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succeq$.

A function $u(.)$ is (strictly) quasiconcave, iff

$$u(\alpha x + (1-\alpha)y) \geq (>) Min[u(x), u(y)]$$

for all $x, y \in X, x \neq y$ for any $\alpha \in (0,1)$

Proposition: If $\succeq$ is rational and (strictly) convex, then any function representing $\succeq$ is (strictly) quasiconcave.

Convexity of preferences does not imply concavity of utility function, only quasiconcavity.

3. Utility Maximization

p : vector of prices for all goods. $p >> 0$

w : wealth of the consumer

Budget set: $B_{p,w} = \{x \in X : p.x \leq w\}$, with $p.x = p_1x_1 + p_2x_2 + ...p_Lx_L$

Utility Maximization Problem (UMP): Which consumption bundle is chosen by the consumer, if he is endowed with w and he faces the price-vector p?

$$\begin{aligned} & \max_{x \in X} u(x) \\ s.t \quad : \quad & p.x \leq w \end{aligned}$$

Note: Price taking behavior assumed

Proposition: If $p >> 0$ and $u(.)$ is continuous, then UMP has a solution.

"Proof": Budget set is closed and bounded (i.e. compact). Maximization of a continuous function over a closed and bounded set has a solution.■

The solution to UMP (the "optimal consumption bundle") depends on p and w: Walrasian Demand Correspondence

$$x(p, w) : \mathbb{R}^L_+ \times \mathbb{R}_+ \rightarrow X$$

If $x(p, w)$ single valued: Walrasian Demand Function

Note: $x(p, w)$ is a "vector" of L individual functions, one for each good. $x_l(p, w)$ denotes "component" for good l.

Proposition: Suppose that $u(.)$ is continuous and that the represented preferences are locally nonsatiated. Then the Walrasian demand correspondence fulfills the following properties:

i) $x(p, w) = x(\alpha p, \alpha w)$ for any $\alpha > 0$. (The correspondence is homogenous of degree zero in (p, w))

ii) $p.x = w$ for all $x \in x(p, w)$ (Walras' law)

iii) If $u(x)$ is quasiconcave (i.e. preferences convex), $x(p, w)$ is convex. If $u(.)$ is strictly quasiconcave (i.e. preferences strictly convex), $x(p, w)$ is single valued.

Proof:

i) $\{x \in X : px \leq w\} = \{x \in X : \alpha px \leq \alpha w\}$: Set of feasible bundles unchanged.

ii) Assume to the contrary $p.x < w$. Then there exists a small neighborhood of x, that is also feasible. By nonsatiation, at least one bundle y in this neighborhood for which $y \succ x$.Since y is also feasible, x cannot be utility maximizing - Contradiction.

iii) Suppose $u(x)$ is quasiconcave. If $x(p, w)$ is single valued, it is also convex. If $x(p, w)$ is not single valued, then denote by x and x"two different optimal bundles. Since both are optimal $u(x) = u(x'')$. By quasiconcavity $u(\alpha x + (1-\alpha)x'') \geq Min[u(x), u(x'')] = u(x)$ for any $\alpha \in (0, 1)$. Since x is optimal, it must hold that $u(\alpha x + (1-\alpha)x'') = u(x)$. Hence, $\alpha x + (1-\alpha)x''$ is also optimal. Suppose $u(x)$ is strictly quasiconcave, and $x(p, w)$ is not single-valued. This implies $u(\alpha x + (1-\alpha)x'') > Min[u(x), u(x'')] = u(x)$. But then x is not optimal - Contradiction$\blacksquare$

Definition: Given $A \subset \mathbb{R}^m$ and a closed set $Y \subset \mathbb{R}^K$, the correspondence $\varphi : A \to Y$ has a closed graph, if for any two sequences $x^n \to \overline{x} \in A$ with $y^n \to \overline{y}$, with $y^n \in \varphi(x^n)$ for all n, we have $\overline{y} \in \varphi(\overline{x})$.

Definition: Given $A \subset \mathbb{R}^m$ and a closed set $Y \subset \mathbb{R}^K$, the correspondence $\varphi : A \to Y$ is upper hemicontinuous (uhc), if

i) it has a closed graph.

ii) for every compact set $B \subset A$, the set $\varphi(B) = \{y \in Y : y \in \varphi(x)$ for some x$\in B\}$ is bounded.

Proposition: If a single-valued correspondence is uhc, it is continuous.

Proposition: Suppose a continuous utility function representing locally non-satiated preferences on the consumption set $X \subset \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ is uhc at all $(p, w) >> 0$.

Indirect utility function $v(p, w) : \mathbb{R}^L_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ gives the maximum utility, which is attainable for price vector p and wealth w.

Let $x^*(p, w)$ be an element of $x(p, w)$. Then $v(p, w) = u(x^*(p, w))$.

For given $\succeq$, $v(p, w)$ is not unique.

Proposition: Suppose that $u(.)$ is continuous and that the represented preferences are locally nonsatiated. Then the indirect utility function is:

i) homogenous of degree zero in (p, w).

ii) strictly increasing in w and strictly nonincreasing in p_l for any l.

iii) quasiconvex, i.e: the set $\{(p, w) : v(p, w) \leq \overline{v}\}$ is convex for any $\overline{v}$.

iv) continuous in p and w.

4. Expenditure Minimization Problem

Which consumption bundle minimizes consumer's expenditures, if he faces the price vector p and he has to achieve a utility level of u?

$$\begin{aligned} &\min_{x \in X} p.x \\ s.t \quad : \quad & u(x) \geq u \end{aligned}$$

EMP "dual" of UMP.

Normalize the utility function such that $u(0) = 0$

Proposition: Suppose that $u(.)$ is continuous, that the represented preferences are defined on $\mathbb{R}^L_+$ and locally nonsatiated, and that $p >> 0$. Then:

i) If x^* is optimal in the UMP with $w >> 0$, then x^* is also a solution of the EMP with a required utility of $u(x^*)$. Moreover, the minimized expenditures are w.

ii) If x^* is optimal in the EMP with a required utility $u > 0$, then x^* is also a solution of the UMP when wealth is px^*. Moreover, the maximized utility of the UMP is exactly u.

For $p >> 0$ and $u > 0$, the value of the EMP is denoted as "expenditure function" $e(p, u)$.

Proposition: Suppose that $u(.)$ is continuous, and that the represented preferences are defined on $\mathbb{R}_+^L$ and locally nonsatiated. Then it holds:

i) $e(\alpha p, u) = \alpha e(p, u)$ for any $\alpha > 0$. ($e(p, u)$ is homogenous of degree 1 in p)

ii) $e(p, u)$ strictly increasing in u and strictly nondecreasing in p_l for any l.

iii) concave in p.

iv) continuous in p and u.

The solution to EMP depends on p and u and is called the Hicksian (or Compensated) Demand Correspondence

$$h(p, u) : \mathbb{R}^L_+ \times \mathbb{R}_+ \to X$$

Hicksian Demand also called "compensated", because in case of price increase it gives the demand which occurs if the consumer is "compensated" for the increasing price with an increase in wealth, so that the attainable utility remains the same. A change in the Hicksian demand due to a price change is only induced by the change in relative prices between the goods, but not by a change in the "real wealth".

Note:

$h(p, u)$ is a "vector" of L individual correspondences, one for each good. $h_l(p, u)$ denotes "component" for good l.

$h(p, u) = x(p, e(p, u))$ and $x(p, w) = h(p, v(p, w))$

Proposition: Suppose that $u(.)$ is continuous, that the represented preferences are defined on $\mathbb{R}^L_+$ and locally nonsatiated, and that $p >> 0$. Then the Hicksian demand correspondence has the following properties:

i) $h(p, u) = h(\alpha p, u)$ for any $\alpha > 0$. (The correspondence is homogenous of degree zero in p)

ii) For any $x \in h(p, u), u(x) = u$

iii) If $u(x)$ is quasiconcave (i.e. preferences convex), $h(p, u)$ is convex. If $u(.)$ is strictly quasiconcave (i.e. preferences strictly convex), $h(p, u)$ is single valued.

5. Relation between Demand, Indirect Utility, and Expenditure Functions

Proposition: Suppose that $u(.)$ is continuous and strictly quasiconcave, and that the represented preferences are defined on $\mathbb{R}^L_+$ and locally nonsatiated. Then it holds that:

$$h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l} \text{ for all } l = 1, 2...L.$$

Proposition (Slutsky equation): Suppose that $u(.)$ is continuous and strictly quasiconcave, and that the represented preferences are defined on $\mathbb{R}^L_+$ and locally nonsatiated. Then it holds for all $k, l = 1, 2...L$:

$$\frac{\partial x_l(p, w)}{\partial p_k} = \frac{\partial h_l(p, u)}{\partial p_k} - \frac{\partial x_l(p, w)}{\partial w} x_k(p, w).$$

Proof:
Consider a consumer facing $(\overline{p}, \overline{w})$ and attaining $\overline{u}$. Since $h(p,u) = x(p, e(p,u))$, it holds that

$$\frac{\partial h_l(\overline{p}, \overline{w})}{\partial p_k} = \frac{\partial x_l(\overline{p}, e(\overline{p}, \overline{u}))}{\partial p_k} + \frac{\partial x_l(\overline{p}, e(\overline{p}, \overline{u}))}{\partial w} \frac{\partial e(\overline{p}, \overline{u})}{\partial p_k}$$

Since $h_l(p,u) = \frac{\partial e(p,u)}{\partial p_l}$, we get

$$\frac{\partial h_l(\overline{p}, \overline{w})}{\partial p_k} = \frac{\partial x_l(\overline{p}, e(\overline{p}, \overline{u}))}{\partial p_k} + \frac{\partial x_l(\overline{p}, e(\overline{p}, \overline{u}))}{\partial w} h_k(\overline{p}, \overline{u})$$

Since $\overline{w} = e(\overline{p}, \overline{u})$ and $h_k(\overline{p}, \overline{u}) = x_k(\overline{p}, e(\overline{p}, \overline{u})) = x_k(\overline{p}, \overline{w})$, we have

$$\frac{\partial h_l(\overline{p}, \overline{w})}{\partial p_k} = \frac{\partial x_l(\overline{p}, \overline{w})}{\partial p_k} + \frac{\partial x_l(\overline{p}, \overline{w})}{\partial w} x_k(\overline{p}, \overline{u})$$

■

The Slutsky equation shows that the overall effect of a price change on Walrasian demand consists of two parts: Substitution and Income Effect.

Proposition (Roy's identity): Suppose that $u(.)$ is continuous and strictly quasiconcave, and that the represented preferences are defined on $\mathbb{R}^L_+$ and locally nonsatiated. Furthermore, suppose that the indirect utility function is differentiable at $(\overline{p}, \overline{w}) >> 0$. Then it holds for all $l = 1, 2...L$:

$$x_l(\overline{p}, \overline{w}) = -\frac{\partial v(\overline{p}, \overline{w})/\partial p_l}{\partial v(\overline{p}, \overline{w})/\partial w}.$$

6. Welfare Evaluation

What is the impact of a price change $p^0 \rightarrow p^1$ on the well-being of a consumer?

Consumer better off iff $v(p^0, w) < v(p^1, w)$ for any $v(.)$ derived from $\succeq$.

Money metric indirect utility function:
For any given fixed $\overline{p} >> 0$, $e(\overline{p}, v(p, w))$ - viewed as function in (p, w) - is an indirect utility function

Welfare change in money terms: $e(\overline{p}, v(p^1, w)) - e(\overline{p}, v(p^0, w))$

Choice of $\overline{p} >> 0$:

Equivalent variation: What is the required change in wealth, to get the same utility with the initial price system, as the consumer gets with the new prices and unchanged wealth:

$$EV(p^0, p^1, w) = e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = e(p^0, u^1) - w$$

Compensating variation: What is the required change in wealth, to compensate for the change in prices:

$$CV(p^0, p^1, w) = e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = w - e(p^1, u^0)$$

Welfare analysis with partial information:

Consumer's expenditure function unknown

Proposition: Suppose consumer's preferences are rational and locally non-satiated. If $p^1 x^0 < p^0 x^0$, then the consumer is strictly better off under p^1 than under p^0.

Proposition: Suppose that the consumer has a differentiable expenditure function, and $p^1 x^0 > p^0 x^0$. Then there exists $\overline{\alpha} \in (0, 1)$ such that for all $\alpha < \overline{\alpha}$ the consumer is strictly better off under p^0 then under $\alpha p^1 + (1 - \alpha) p^0$.

7. Revealed preferences

Till now: Preferences $\longrightarrow$ choice (i.e. demand function)

Now: What do observed choices reveal about preferences?

Definition: $x(p, w)$ satisfies the weak axiom of revealed preference, if for any two situations (p, w) and (p', w') it holds:

$$\begin{aligned} px(p', w') &\leq w \text{ and } x(p, w) \neq x(p', w') \Rightarrow \\ p'x(p, w) &> w' \end{aligned}$$

Definition: $x(p, w)$ satisfies the strong axiom of revealed preference, if for any list $(p^1, w^1), (p^2, w^2)...(p^N, w^N)$ with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n < N$ it holds:

$$\begin{aligned} p^n x(p^{n+1}, w^{n+1}) &\leq w^n \text{ for all } n < N \Rightarrow \\ p^N x(p^1, w^1) &> w^N \end{aligned}$$

Proposition: If $x(p, w)$ satisfies the strong axiom of revealed preferences, then there exist a rational preference relation from which $x(p, w)$ can be derived.